Appendix A

Answers to Problems

A.1 Problems from section 2.2.4

7. (a)

$$\dfrac{\dfrac{\overline{A}^{\,x}}{A \to A}\to \mathsf{I}_x}{}$$

$$\dfrac{\dfrac{\dfrac{\overline{A}^{\,x}}{A \to A}\to \mathsf{I}_x \quad \dfrac{\overline{A}^{\,y}}{}}{A}\to \mathsf{E}}{A \to A}\to \mathsf{I}_y$$

(b)

$$\dfrac{\dfrac{\dfrac{\overline{A \wedge A}^{\,p}}{A}\wedge \mathsf{E}^{\mathsf{L}}}{A \wedge A \to A}\to \mathsf{I}_p}{}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \wedge A}^{\,p}}{A}\wedge \mathsf{E}^{\mathsf{R}} \quad \dfrac{\overline{A \wedge A}^{\,p}}{A}\wedge \mathsf{E}^{\mathsf{R}}}{A}\wedge \mathsf{I}}{A}\wedge \mathsf{E}^{\mathsf{L}}}{A \wedge A \to A}\to \mathsf{I}_p$$

(c)

$$\dfrac{\dfrac{}{A \vee B}\,d \qquad \dfrac{\dfrac{\overline{A}\,x}{B \vee A}\vee \mathsf{I}^{\mathsf{R}} \qquad \dfrac{\overline{B}\,y}{B \vee A}\vee \mathsf{I}^{\mathsf{L}}}{B \vee A}\vee \mathsf{E}_{x,y}}{A \vee B \to B \vee A}\to \mathsf{I}_d$$

$$\dfrac{\dfrac{}{A \vee B}\,d \qquad \dfrac{\dfrac{\dfrac{\overline{A}\,x}{A \vee B}\vee \mathsf{I}^{\mathsf{L}} \qquad \dfrac{\overline{A}\,x'}{B \vee A}\vee \mathsf{I}^{\mathsf{R}} \qquad \dfrac{\overline{B}\,y'}{B \vee A}\vee \mathsf{I}^{\mathsf{L}}}{B \vee A}\vee \mathsf{E}_{x',y'} \qquad \dfrac{\dfrac{\overline{B}\,y}{A \vee B}\vee \mathsf{I}^{\mathsf{R}} \qquad \dfrac{\overline{A}\,x''}{B \vee A}\vee \mathsf{I}^{\mathsf{R}} \qquad \dfrac{\overline{B}\,y''}{B \vee A}\vee \mathsf{I}^{\mathsf{L}}}{B \vee A}\vee \mathsf{E}_{x'',y''}}{B \vee A}\vee \mathsf{E}_{x,y}}{A \vee B \to B \vee A}\to \mathsf{I}_d$$

8. (a)

$$\dfrac{\dfrac{\overline{A}\,x}{B \to A}\to \mathsf{I}_y}{A \to B \to A}\to \mathsf{I}_x$$

(b)

$$\dfrac{\dfrac{}{A \vee A}\,d \qquad \dfrac{\overline{A}\,x \qquad \overline{A}\,y}{A}\vee \mathsf{E}_{x,y}}{A \vee A \to A}\to \mathsf{I}_d$$

(c)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B}\,p}{A}\wedge \mathsf{E}^{\mathsf{L}}}{A \vee B}\vee \mathsf{I}^{\mathsf{L}}}{A \wedge B \to A \vee B}\to \mathsf{I}_p$$

(d)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B}\,f \qquad \dfrac{\overline{A}\,x}{}}{B}\to \mathsf{E}}{(A \to B) \to B}\to \mathsf{I}_f}{A \to (A \to B) \to B}\to \mathsf{I}_x$$

(e)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B \to C}\,f \quad \dfrac{}{A}\,y}{B \to C}\,{\to}\mathsf{E} \quad \dfrac{}{B}\,x}{C}\,{\to}\mathsf{E}}{A \to C}\,{\to}\mathsf{I}_y}{B \to A \to C}\,{\to}\mathsf{I}_x}{(A \to B \to C) \to B \to A \to C}\,{\to}\mathsf{I}_f$$

(f)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B \to C}\,f \quad \dfrac{\dfrac{}{A}\,x \quad \dfrac{}{B}\,y}{A \wedge B}\,{\wedge}\mathsf{I}}{C}\,{\to}\mathsf{E}}{B \to C}\,{\to}\mathsf{I}_y}{A \to B \to C}\,{\to}\mathsf{I}_x}{(A \wedge B \to C) \to A \to B \to C}\,{\to}\mathsf{I}_f$$

(g)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B \to C}\,f \quad \dfrac{\dfrac{}{A \wedge B}\,p}{A}\,{\wedge}\mathsf{E}^{\mathsf{L}}}{B \to C}\,{\to}\mathsf{E} \quad \dfrac{\dfrac{}{A \wedge B}\,p}{B}\,{\wedge}\mathsf{E}^{\mathsf{R}}}{C}\,{\to}\mathsf{E}}{A \wedge B \to C}\,{\to}\mathsf{I}_p}{(A \to B \to C) \to A \wedge B \to C}\,{\to}\mathsf{I}_f$$

(h)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B}\,f \quad \dfrac{}{A}\,x}{B}\,{\to}\mathsf{E} \quad \dfrac{\dfrac{}{A \to C}\,g \quad \dfrac{}{A}\,x}{C}\,{\to}\mathsf{E}}{B \wedge C}\,{\wedge}\mathsf{I}}{A \to B \wedge C}\,{\to}\mathsf{I}_x}{(A \to C) \to A \to B \wedge C}\,{\to}\mathsf{I}_g}{(A \to B) \to (A \to C) \to A \to B \wedge C}\,{\to}\mathsf{I}_f$$

(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B}\,f \quad \dfrac{\dfrac{}{A \wedge C}\,p}{A}\,\wedge\mathsf{E}^{\mathsf{L}}}{B}\,\to\mathsf{E} \quad \dfrac{\dfrac{\dfrac{}{C \to D}\,g \quad \dfrac{\dfrac{}{A \wedge C}\,p}{C}\,\wedge\mathsf{E}^{\mathsf{R}}}{D}\,\to\mathsf{E}}{B \wedge D}\,\wedge\mathsf{I}}{A \wedge C \to B \wedge D}\,\to\mathsf{I}_p}{(C \to D) \to A \wedge C \to B \wedge D}\,\to\mathsf{I}_g}{(A \to B) \to (C \to D) \to A \wedge C \to B \wedge D}\,\to\mathsf{I}_f$$

(j)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \to C}\,g \quad \dfrac{\dfrac{}{A \to B}\,f \quad \dfrac{}{A}\,x}{B}\,\to\mathsf{E}}{C}\,\to\mathsf{E}}{A \to C}\,\to\mathsf{I}_x}{(B \to C) \to A \to C}\,\to\mathsf{I}_g}{(A \to B) \to (B \to C) \to A \to C}\,\to\mathsf{I}_f$$

(k)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B \to C}\,f \quad \dfrac{}{A}\,x}{B \to C}\,\to\mathsf{E} \quad \dfrac{\dfrac{}{A \to B}\,g \quad \dfrac{}{A}\,x}{B}\,\to\mathsf{E}}{C}\,\to\mathsf{E}}{A \to C}\,\to\mathsf{I}_x}{(A \to B) \to A \to C}\,\to\mathsf{I}_g}{(A \to B \to C) \to (A \to B) \to A \to C}\,\to\mathsf{I}_f$$

(l)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge (B \vee C)}\,p}{B \vee C}\,\wedge\mathsf{E}^{\mathsf{R}} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge (B \vee C)}\,p}{A}\,\wedge\mathsf{E}^{\mathsf{L}} \quad \dfrac{}{B}\,x}{A \wedge B}\,\wedge\mathsf{I}}{(A \wedge B) \vee (A \wedge C)}\,\vee\mathsf{I}^{\mathsf{L}} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge (B \vee C)}\,p}{A}\,\wedge\mathsf{E}^{\mathsf{L}} \quad \dfrac{}{C}\,y}{A \wedge C}\,\wedge\mathsf{I}}{(A \wedge B) \vee (A \wedge C)}\,\vee\mathsf{I}^{\mathsf{R}}}{(A \wedge B) \vee (A \wedge C)}\,\vee\mathsf{E}_{x,y}}{A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)}\,\to\mathsf{I}_p$$

(m)

$$\dfrac{\dfrac{}{(A\wedge B)\vee(A\wedge C)}\,d \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A\wedge B}\,p}{A}\wedge\mathsf{E}^{\mathsf{L}} \quad \dfrac{\dfrac{\dfrac{}{A\wedge B}\,p}{B}\wedge\mathsf{E}^{\mathsf{R}}}{B\vee C}\vee\mathsf{I}^{\mathsf{L}}}{A\wedge(B\vee C)}\wedge\mathsf{I} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A\wedge C}\,q}{A}\wedge\mathsf{E}^{\mathsf{L}} \quad \dfrac{\dfrac{\dfrac{}{A\wedge C}\,q}{C}\wedge\mathsf{E}^{\mathsf{R}}}{B\vee C}\vee\mathsf{I}^{\mathsf{R}}}{A\wedge(B\vee C)}\wedge\mathsf{I}}{A\wedge(B\vee C)}\vee\mathsf{E}_{p,q}}{A\wedge(B\vee C)}\to\mathsf{I}_d}{(A\wedge B)\vee(A\wedge C)\to A\wedge(B\vee C)}$$

(n)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A\to\bot}\,f \quad \dfrac{}{A}\,x}{\bot}\to\mathsf{E}}{(A\to\bot)\to\bot}\to\mathsf{I}_f}{A\to(A\to\bot)\to\bot}\to\mathsf{I}_x}{A\to\neg\neg A}=$$

(o)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A\to\bot)\vee(B\to\bot)}\,d \qquad \dfrac{\dfrac{}{A\to\bot}\,f \quad \dfrac{\dfrac{}{A\wedge B}\,p}{A}\wedge\mathsf{E}^{\mathsf{L}}}{\bot}\to\mathsf{E} \qquad \dfrac{\dfrac{}{B\to\bot}\,g \quad \dfrac{\dfrac{}{A\wedge B}\,p}{B}\wedge\mathsf{E}^{\mathsf{R}}}{\bot}\to\mathsf{E}}{\bot}\vee\mathsf{E}_{f,g}}{\bot}\to\mathsf{I}_d}{(A\to\bot)\vee(B\to\bot)\to\bot}\to\mathsf{I}_p}{A\wedge B\to(B\to\bot)\vee(B\to\bot)\to\bot}=}{A\wedge B\to\neg(\neg A\vee\neg B)}$$

(p)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A\vee B}\,d \qquad \dfrac{\dfrac{\dfrac{}{(A\to\bot)\wedge(B\to\bot)}\,p}{A\to\bot}\wedge\mathsf{E}^{\mathsf{L}} \quad \dfrac{}{A}\,x}{\bot}\to\mathsf{E} \qquad \dfrac{\dfrac{}{(A\to\bot)\wedge(B\to\bot)}\,p \quad \dfrac{}{B}\,y}{\bot}\to\mathsf{E}}{\bot}\vee\mathsf{E}_{x,y}}{\bot}\to\mathsf{I}_p}{(A\to\bot)\wedge(B\to\bot)\to\bot}\to\mathsf{I}_d}{A\vee B\to(A\to\bot)\wedge(B\to\bot)\to\bot}=}{A\vee B\to\neg(\neg A\wedge\neg B)}$$

(q)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \to A \to C \wedge A}\,g \qquad \dfrac{\dfrac{\dfrac{}{A \to B \wedge A}\,f \qquad \dfrac{}{A}\,x}{B \wedge A}\,{\to}\mathsf{E}}{B}\,{\wedge}\mathsf{E}^{\mathsf{L}}}{A \to C \wedge A}\,{\to}\mathsf{E} \qquad \dfrac{\dfrac{\dfrac{}{A \to B \wedge A}\,f \qquad \dfrac{}{A}\,x}{B \wedge A}\,{\to}\mathsf{E}}{A}\,{\wedge}\mathsf{E}^{\mathsf{R}}}{C \wedge A}\,{\to}\mathsf{E}}{A \to C \wedge A}\,{\to}\mathsf{I}_x}{(B \to A \to C \wedge A) \to A \to C \wedge A}\,{\to}\mathsf{I}_g}{(A \to B \wedge A) \to (B \to A \to C \wedge A) \to A \to C \wedge A}\,{\to}\mathsf{I}_f$$

9. (a)

$$\dfrac{\dfrac{\mathsf{X}}{\dfrac{A}{A \vee (A \to \bot)}\,{\vee}\mathsf{I}^{\mathsf{L}}}\,=}{A \vee \neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A}\,x}{\mathsf{X}}}{\dfrac{\bot}{A \to \bot}\,{\to}\mathsf{I}_x}\,{\vee}\mathsf{I}^{\mathsf{R}}}{A \vee (A \to \bot)}\,=}{A \vee \neg A}$$

(b)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \to \bot) \vee (B \to \bot) \to \bot}\, f \qquad \dfrac{\dfrac{\dfrac{}{A}\, x}{\vdots}}{\dfrac{\bot}{A \to \bot} \to \mathsf{I}_x}}{(A \to \bot) \vee (B \to \bot)} \vee \mathsf{I}^{\mathsf{L}}}{\bot} \to \mathsf{E}}{A \wedge B} \bot \mathsf{E}}{((A \to \bot) \vee (B \to \bot) \to \bot) \to A \wedge B} \to \mathsf{I}_f}{\neg(\neg A \vee \neg B) \to A \wedge B} =$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \to \bot) \vee (B \to \bot) \to \bot}\, f \qquad \dfrac{\dfrac{\dfrac{}{B}\, y}{\vdots}}{\dfrac{\bot}{B \to \bot} \to \mathsf{I}_y}}{(A \to \bot) \vee (B \to \bot)} \vee \mathsf{I}^{\mathsf{R}}}{\bot} \to \mathsf{E}}{A \wedge B} \bot \mathsf{E}}{((A \to \bot) \vee (B \to \bot) \to \bot) \to A \wedge B} \to \mathsf{I}_f}{\neg(\neg A \vee \neg B) \to A \wedge B} =$$

(c)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \to \bot) \wedge (B \to \bot) \to \bot}\, f \qquad \dfrac{\dfrac{\dfrac{}{A}\, x}{\vdots}}{\dfrac{\bot}{A \to \bot} \to \mathsf{I}_x} \qquad \dfrac{\dfrac{\dfrac{}{B}\, y}{\vdots}}{\dfrac{\bot}{B \to \bot} \to \mathsf{I}_y}}{(A \to \bot) \wedge (B \to \bot)} \wedge \mathsf{I}}{\bot} \to \mathsf{E}}{A \vee B} \bot \mathsf{E}}{((A \to \bot) \wedge (B \to \bot) \to \bot) \to A \vee B} \to \mathsf{I}_f}{\neg(\neg A \wedge \neg B) \to A \vee B} =$$

(d)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{f}
\UnaryInfC{$(A \to \bot) \to \bot$}
\AxiomC{}
\RightLabel{x}
\UnaryInfC{A}
\noLine
\UnaryInfC{X}
\noLine
\UnaryInfC{$\bot$}
\RightLabel{$\to\mathsf{I}_x$}
\UnaryInfC{$A \to \bot$}
\RightLabel{$\to\mathsf{E}$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot\mathsf{E}$}
\UnaryInfC{A}
\RightLabel{$\to\mathsf{I}_f$}
\UnaryInfC{$((A \to \bot) \to \bot) \to A$}
\RightLabel{$=$}
\UnaryInfC{$\neg\neg A \to A$}
\end{prooftree}
$$

10. (a) Unprovable

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{d}
\UnaryInfC{$A \vee B$}
\AxiomC{}
\RightLabel{x}
\UnaryInfC{A}
\AxiomC{X}
\noLine
\UnaryInfC{B}
\RightLabel{$\wedge\mathsf{I}$}
\BinaryInfC{$A \wedge B$}
\AxiomC{X}
\noLine
\UnaryInfC{A}
\AxiomC{}
\RightLabel{y}
\UnaryInfC{B}
\BinaryInfC{$A \wedge B$}
\RightLabel{$\vee\mathsf{E}_{x,y}$}
\TrinaryInfC{$A \wedge B$}
\RightLabel{$\to\mathsf{I}_d$}
\UnaryInfC{$A \vee B \to A \wedge B$}
\end{prooftree}
$$

(b) Provable

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{g}
\UnaryInfC{$B \to \bot$}
\AxiomC{}
\RightLabel{f}
\UnaryInfC{$A \to B$}
\AxiomC{}
\RightLabel{x}
\UnaryInfC{A}
\RightLabel{$\to\mathsf{E}$}
\BinaryInfC{B}
\RightLabel{$\to\mathsf{E}$}
\BinaryInfC{$\bot$}
\RightLabel{$\to\mathsf{I}_x$}
\UnaryInfC{$A \to \bot$}
\RightLabel{$\to\mathsf{I}_g$}
\UnaryInfC{$(B \to \bot) \to A \to \bot$}
\RightLabel{$\to\mathsf{I}_f$}
\UnaryInfC{$(A \to B) \to (B \to \bot) \to A \to \bot$}
\RightLabel{$=$}
\UnaryInfC{$(A \to B) \to \neg B \to \neg A$}
\end{prooftree}
$$

(c) Unprovable

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(B \to \bot) \to A \to \bot}\, f \quad \dfrac{\dfrac{\bot}{B \to \bot}\, \to \mathsf{I}_y}{A \to \bot}\, \to \mathsf{E} \quad \dfrac{}{A}\, x}{\bot}\, \to \mathsf{E}}{B}\, \bot\mathsf{E}}{A \to B}\, \to \mathsf{I}_x}{((B \to \bot) \to A \to \bot) \to A \to B}\, \to \mathsf{I}_f}{(\neg B \to \neg A) \to A \to B}\, =$$

(d) Unprovable

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A}\, x}{\bot}\, \mathsf{X}}{A \to \bot}\, \to \mathsf{I}_x}{(A \to \bot) \vee B}\, \vee \mathsf{I}^{\mathsf{L}}}{(A \to B) \to (A \to \bot) \vee B}\, \to \mathsf{I}_f \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \to B}\, f \quad \dfrac{}{A}\, x}{B}\, \to \mathsf{E}}{\bot}\, \mathsf{X}}{A \to \bot}\, \to \mathsf{I}_x}{(A \to \bot) \vee B}\, \vee \mathsf{I}^{\mathsf{L}}}{(A \to B) \to (A \to \bot) \vee B}\, \to \mathsf{I}_f$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B}\, f \quad A}{B}\, \to \mathsf{E}}{(A \to \bot) \vee B}\, \vee \mathsf{I}^{\mathsf{R}}}{(A \to B) \to (A \to \bot) \vee B}\, \to \mathsf{I}_f}{(A \to B) \to \neg A \vee B}\, =$$

(e) Provable

$$\dfrac{\dfrac{\dfrac{}{(A \to \bot) \vee B}\,d \qquad \dfrac{\dfrac{\dfrac{}{A \to \bot}\,f \quad \dfrac{}{A}\,x}{\bot}\,\to\mathsf{E}}{B}\,\bot\mathsf{E} \qquad \dfrac{}{B}\,y}{B}\,\vee\mathsf{E}_{f,y}}{A \to B}\,\to\mathsf{I}_x}{((A \to \bot) \vee B) \to A \to B}\,\to\mathsf{I}_d}{\neg A \vee B \to A \to B}\,=$$

11. (a)

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{P}\\ A \wedge B\end{array}}{A}\,\wedge\mathsf{E}^{\mathsf{L}}$$

(b)

$$\begin{array}{l}\vdots\,\mathcal{A}\\ A\end{array}$$

(c)

$$\begin{array}{l}\vdots\,\mathcal{B}\\ B\end{array}$$

12. A verification is the same as a proof that cannot be normalized in any way.

13. They can all be applied freely to available hypotheses until no elimination can be done, and then either a proof for the current goal can just be looked up from these (potentially incomplete) proofs, or the proof gets stuck at that goal.

A.2 Problems from section 2.3.7 #14

The following are answers to problem 2.3.7 #14, which asks for the proofs of 2.2.4 only with proof terms.

7. (a)

$$\dfrac{\dfrac{}{x : A \vdash x : A}\,\mathsf{hyp}}{\vdash \lambda x.x : A \to A}\,{\to}\mathsf{I}_x \qquad \dfrac{\dfrac{\dfrac{}{y : A, x : A \vdash x : A}\,\mathsf{hyp}}{y : A \vdash \lambda x.x : A \to A}\,{\to}\mathsf{I}_x \quad \dfrac{}{y : A \vdash y : A}\,\mathsf{hyp}}{y : A \vdash (\lambda x.x)\, y : A}\,{\to}\mathsf{E}}{\vdash \lambda y.\ (\lambda x.x)\, y : A \to A}\,{\to}\mathsf{I}_y$$

(b)

$$\dfrac{\dfrac{\dfrac{}{p : A \wedge A \vdash p : A \wedge A}\,\mathsf{hyp}}{p : A \wedge A \vdash \mathsf{fst}\ p : A}\,{\wedge}\mathsf{E}^{\mathsf{L}}}{\vdash \lambda p.\ \mathsf{fst}\ p : A \wedge A \to A}\,{\to}\mathsf{I}_p \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p : A \wedge A \vdash p : A \wedge A}\,\mathsf{hyp}}{p : A \wedge A \vdash \mathsf{snd}\ p : A}\,{\wedge}\mathsf{E}^{\mathsf{R}} \quad \dfrac{\dfrac{}{p : A \wedge A \vdash p : A \wedge A}\,\mathsf{hyp}}{p : A \wedge A \vdash \mathsf{snd}\ p : A}\,{\wedge}\mathsf{E}^{\mathsf{R}}}{p : A \wedge A \vdash \langle \mathsf{snd}\ p\,, \mathsf{snd}\ p \rangle : A \wedge A}\,{\wedge}\mathsf{I}}{p : A \wedge A \vdash \mathsf{fst}\ \langle \mathsf{snd}\ p\,, \mathsf{snd}\ p \rangle : A}\,{\wedge}\mathsf{E}^{\mathsf{L}}}{\vdash \lambda p.\ \mathsf{fst}\ \langle \mathsf{snd}\ p\,, \mathsf{snd}\ p \rangle : A \wedge A \to A}\,{\to}\mathsf{I}_p$$

(c)

$$\dfrac{\dfrac{\dfrac{}{d : A \vee B \vdash d : A \vee B}\,\mathsf{hyp} \quad \dfrac{\dfrac{}{d : A \vee B, x : A \vdash x : A}\,\mathsf{hyp}}{d : A \vee B, x : A \vdash \mathsf{inr}\ x : B \vee A}\,{\vee}\mathsf{I}^{\mathsf{R}} \quad \dfrac{\dfrac{}{d : A \vee B, y : B \vdash y : B}\,\mathsf{hyp}}{d : A \vee B, y : B \vdash \mathsf{inl}\ y : B \vee A}\,{\vee}\mathsf{I}^{\mathsf{R}}}{d : A \vee V \vdash \mathsf{case}\ d\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathsf{inr}\ x\ ;\ \mathsf{inr}\ y \mapsto \mathsf{inl}\ y\ \} : B \vee A}\,{\vee}\mathsf{E}_{x,y}}{\vdash \lambda d.\ \mathsf{case}\ d\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathsf{inr}\ x\ ;\ \mathsf{inr}\ y \mapsto \mathsf{inl}\ y\ \} : A \vee B \to B \vee A}\,{\to}\mathsf{I}_d$$

Version with detours omitted for space.

8. (a)

$$\dfrac{\dfrac{\dfrac{}{x : A,\ y : B \vdash x : A}\,\mathsf{hyp}}{x : A \vdash \lambda y.x : B \to A}\,{\to}\mathsf{I}_y}{\vdash \lambda x.\lambda y.x : A \to B \to A}\,{\to}\mathsf{I}_x$$

(b)

$$\dfrac{\dfrac{}{d : A \vee A \vdash d : A \vee A}\,\mathsf{hyp} \quad \dfrac{}{d : A \vee A,\ x : A \vdash x : A}\,\mathsf{hyp} \quad \dfrac{}{d : A \vee A,\ y : A \vdash y : A}\,\mathsf{hyp}}{d : A \vee A \vdash \mathsf{case}\ d\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto x\ ;\ \mathsf{inr}\ y \mapsto y\ \} : A}\,{\vee}\mathsf{E}_{x,y}}{\vdash \lambda d.\ \mathsf{case}\ d\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto x\ ;\ \mathsf{inr}\ y \mapsto y\ \} : A \vee A \to A}\,{\to}\mathsf{I}_d$$

(c)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p : A \wedge B \vdash p : A \wedge B}\,\text{hyp}}{p : A \wedge B \vdash \mathsf{fst}\ p : A}\,\wedge\mathsf{E}^{\mathsf{L}}}{p : A \wedge B \vdash \mathsf{inl}\ (\mathsf{fst}\ p) : A \vee B}\,\vee\mathsf{I}^{\mathsf{L}}}{\vdash \lambda p.\ \mathsf{inl}\ (\mathsf{fst}\ p) : A \wedge B \to A \vee B}\,\to\mathsf{I}_p$$

(d)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : A,\ f : A \to B \vdash f : A \to B}\,\text{hyp} \qquad \dfrac{}{x : A,\ f : A \to B \vdash x : A}\,\text{hyp}}{x : A,\ f : A \to B \vdash f\ x : B}\,\to\mathsf{E}}{x : A \vdash \lambda f.\ f\ x : (A \to B) \to B}\,\to\mathsf{I}_f}{\vdash \lambda x.\lambda f.\ f\ x : A \to (A \to B) \to B}\,\to\mathsf{I}_x$$

(e)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f : A \to B \to C}\,\text{hyp} \qquad \dfrac{}{\Gamma \vdash y : A}\,\text{hyp}}{\Gamma \vdash f\ y : B \to C}\,\to\mathsf{E} \qquad \dfrac{}{\Gamma \vdash x : B}\,\text{hyp}}{\Gamma = f : A \to B \to C,\ x : B,\ y : A \vdash f\ y\ x : C}\,\to\mathsf{E}}{f : A \to B \to C,\ x : B \vdash \lambda y.\ f\ y\ x : A \to C}\,\to\mathsf{I}_y}{f : A \to B \to \vdash \lambda x.\lambda y.\ f\ y\ x : B \to A \to C}\,\to\mathsf{I}_x}{\vdash \lambda f.\lambda x.\lambda y.\ f\ y\ x : (A \to B \to C) \to B \to A \to C}\,\to\mathsf{I}_f$$

(f)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f : A \wedge B \to C}\,\text{hyp} \qquad \dfrac{\dfrac{}{\Gamma \vdash x : A}\,\text{hyp} \qquad \dfrac{}{\Gamma \vdash y : B}\,\text{hyp}}{\Gamma \vdash \langle x\,,y\rangle : A \wedge B}\,\wedge\mathsf{I}}{\Gamma = f : A \wedge B \to C,\ x : A,\ y : B \vdash f\ \langle x\,,y\rangle : C}\,\to\mathsf{E}}{f : A \wedge B \to C,\ x : A \vdash \lambda y.\ f\ \langle x\,,y\rangle : B \to C}\,\to\mathsf{I}_y}{f : A \wedge B \to C \vdash \lambda x.\lambda y.\ f\ \langle x\,,y\rangle : A \to B \to C}\,\to\mathsf{I}_x}{\vdash \lambda f.\lambda x.\lambda y.\ f\ \langle x\,,y\rangle : (A \wedge B \to C) \to A \to B \to C}\,\to\mathsf{I}_f$$

(g)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f : A \to B \to C}\,\text{hyp} \quad \dfrac{\dfrac{}{\Gamma \vdash p : A \wedge B}\,\text{hyp}}{\Gamma \vdash \mathsf{fst}\ p : A}\,\wedge\mathsf{E}^{\mathsf{L}}}{\Gamma \vdash f\ (\mathsf{fst}\ p) : B \to C}\,\to\mathsf{E} \quad \dfrac{\dfrac{}{\Gamma \vdash p : A \wedge B}\,\text{hyp}}{\Gamma \vdash \mathsf{snd}\ p : B}\,\wedge\mathsf{E}^{\mathsf{R}}}{\Gamma = f : A \to B \to C,\ p : A \wedge B \vdash f\ (\mathsf{fst}\ p)\ (\mathsf{snd}\ p) : C}\,\to\mathsf{E}}{f : A \to B \to C \vdash \lambda p.\ f\ (\mathsf{fst}\ p)\ (\mathsf{snd}\ p) : A \wedge B \to C}\,\to\mathsf{I}_p}{\vdash \lambda f.\lambda p.\ f\ (\mathsf{fst}\ p)\ (\mathsf{snd}\ p) : (A \to B \to C) \to A \wedge B \to C}\,\to\mathsf{I}_f$$

(h)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f : A \to B}\,\text{hyp} \quad \dfrac{}{\Gamma \vdash x : A}\,\text{hyp}}{\Gamma \vdash f\ x : B}\,\to\mathsf{E} \quad \dfrac{\dfrac{}{\Gamma \vdash g : A \to C}\,\text{hyp} \quad \dfrac{}{\Gamma \vdash x : A}\,\text{hyp}}{\Gamma \vdash g\ x : C}\,\to\mathsf{E}}{\Gamma = f : A \to B,\ g : A \to C,\ x : A \vdash \langle f\ x\,, g\ x\rangle : B \wedge C}\,\wedge\mathsf{I}}{f : A \to B,\ g : A \to C \vdash \lambda x.\ \langle f\ x\,, g\ x\rangle : A \to B \wedge C}\,\to\mathsf{I}_x}{f : A \to B \vdash \lambda g.\lambda x.\ \langle f\ x\,, g\ x\rangle : (A \to C) \to A \to B \wedge C}\,\to\mathsf{I}_g}{\vdash \lambda f.\lambda g.\lambda x.\ \langle f\ x\,, g\ x\rangle : (A \to B) \to (A \to C) \to A \to B \wedge C}\,\to\mathsf{I}_f$$

(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f : A \to B}\,\text{hyp} \quad \dfrac{\dfrac{}{\Gamma \vdash p : A \wedge B}\,\text{hyp}}{\Gamma \vdash \mathsf{fst}\ p : A}\,\wedge\mathsf{E}^{\mathsf{L}}}{\Gamma \vdash f\ (\mathsf{fst}\ p) : B}\,\to\mathsf{E} \quad \dfrac{\dfrac{}{\Gamma \vdash g : C \to D}\,\text{hyp} \quad \dfrac{\dfrac{}{\Gamma \vdash p : A \wedge C}\,\text{hyp}}{\Gamma \vdash \mathsf{snd}\ p : C}\,\wedge\mathsf{E}^{\mathsf{R}}}{\Gamma \vdash g\ (\mathsf{snd}\ p) : D}\,\to\mathsf{E}}{\Gamma = f : A \to B,\ g : C \to D,\ p : A \wedge B \vdash \langle f\ (\mathsf{fst}\ p)\,, g\ (\mathsf{snd}\ p)\rangle : B \wedge D}\,\wedge\mathsf{I}}{f : A \to B,\ g : C \to D \vdash \lambda p.\ \langle f\ (\mathsf{fst}\ p)\,, g\ (\mathsf{snd}\ p)\rangle : A \wedge C \to B \wedge D}\,\to\mathsf{I}_p}{f : A \to B \vdash \lambda g.\lambda p.\ \langle f\ (\mathsf{fst}\ p)\,, g\ (\mathsf{snd}\ p)\rangle : (C \to D) \to A \wedge C \to B \wedge D}\,\to\mathsf{I}_g}{\vdash \lambda f.\lambda g.\lambda p.\ \langle f\ (\mathsf{fst}\ p)\,, g\ (\mathsf{snd}\ p)\rangle : (A \to B) \to (C \to D) \to A \wedge C \to B \wedge D}\,\to\mathsf{I}_f$$

(j)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash g : B \to C}\,\text{hyp} \quad \dfrac{\dfrac{}{\Gamma \vdash f : A \to B}\,\text{hyp} \quad \dfrac{}{\Gamma \vdash x : A}\,\text{hyp}}{\Gamma \vdash f\ x : B}\,\to\mathsf{E}}{\Gamma = f : A \to B,\ g : B \to C, x : A \vdash g\ (f\ x) : C}\,\to\mathsf{E}}{f : A \to B,\ g : B \to C \vdash \lambda x.\ g\ (f\ x) : A \to C}\,\to\mathsf{I}_x}{f : A \to B \vdash \lambda g.\lambda x.\ g\ (f\ x) : (B \to C) \to A \to C}\,\to\mathsf{I}_g}{\vdash \lambda f.\lambda g.\lambda x.\ g\ (f\ x) : (A \to B) \to (B \to C) \to A \to C}\,\to\mathsf{I}_f$$

(k)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f : A \to B \to C}\,\text{hyp} \quad \dfrac{}{\Gamma \vdash x : A}\,\text{hyp}}{\Gamma \vdash f\ x : B \to C}\,{\to}\text{E} \quad \dfrac{\dfrac{}{\Gamma \vdash g : A \to B}\,\text{hyp} \quad \dfrac{}{\Gamma \vdash x : A}\,\text{hyp}}{\Gamma \vdash g\ x : B}\,{\to}\text{E}}{\Gamma = f : A \to B \to C,\ g : A \to B,\ x : A \vdash f\ x\ (g\ x) : C}\,{\to}\text{I}_x}{f : A \to B \to C,\ g : A \to B \vdash \lambda x.\ f\ x\ (g\ x) : A \to C}\,{\to}\text{I}_g}{f : A \to B \to C \vdash \lambda g.\lambda x.\ f\ x\ (g\ x) : (A \to B) \to A \to C}\,{\to}\text{I}_f}{\vdash \lambda f.\lambda g.\lambda x.\ f\ x\ (g\ x) : (A \to B \to C) \to (A \to B) \to A \to C}$$

(l)

$$\text{lemma}_1 = \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\ x : B \vdash p : A \wedge (B \vee C)}\,\text{hyp}}{\Gamma,\ x : B \vdash \mathsf{fst}\ p : A}\,{\wedge}\text{E}^{\text{L}} \quad \dfrac{}{\Gamma,\ x : B \vdash x : B}\,\text{hyp}}{\Gamma,\ x : B \vdash \langle \mathsf{fst}\ p\,,\, x \rangle : A \wedge B}\,{\wedge}\text{I}}{\Gamma,\ x : B \vdash \mathsf{inl}\ \langle \mathsf{fst}\ p\,,\, x \rangle : (A \wedge B) \vee (A \wedge C)}\,{\vee}\text{I}^{\text{L}}$$

$$\text{lemma}_2 = \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\ y : C \vdash p : A \wedge (B \vee C)}\,\text{hyp}}{\Gamma,\ y : C \vdash \mathsf{fst}\ p : A}\,{\wedge}\text{E}^{\text{L}} \quad \dfrac{}{\Gamma,\ y : C \vdash y : C}\,\text{hyp}}{\Gamma,\ y : C \vdash \langle \mathsf{fst}\ p\,,\, y \rangle : A \wedge C}\,{\wedge}\text{I}}{\Gamma,\ y : C \vdash \mathsf{inr}\ \langle \mathsf{fst}\ p\,,\, y \rangle : (A \wedge B) \vee (A \wedge C)}\,{\vee}\text{I}^{\text{R}}$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p : A \wedge (B \vee C)}\,\text{hyp}}{\Gamma \vdash \mathsf{snd}\ p : B \vee C}\,{\wedge}\text{E}^{\text{R}} \qquad \text{lemma}_1 \qquad \text{lemma}_2}{\Gamma = p : A \wedge (B \vee C) \vdash \begin{array}{l}\mathsf{case}\ \mathsf{snd}\ p\ \mathsf{of}\\ \quad \mathsf{inl}\ x \mapsto \mathsf{inl}\ \langle \mathsf{fst}\ p\,,\, x \rangle\\ \quad \mathsf{inr}\ y \mapsto \mathsf{inr}\ \langle \mathsf{fst}\ p\,,\, y \rangle\end{array} : (A \wedge B) \vee (A \wedge C)}\,{\vee}\text{E}_{x,y}$$

$$\vdash \begin{array}{l}\lambda p.\ \mathsf{case}\ \mathsf{snd}\ p\ \mathsf{of}\\ \quad \mathsf{inl}\ x \mapsto \mathsf{inl}\ \langle \mathsf{fst}\ p\,,\, x \rangle\\ \quad \mathsf{inr}\ y \mapsto \mathsf{inr}\ \langle \mathsf{fst}\ p\,,\, y \rangle\end{array} : A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$$

(m)

$$\text{lemma}_1 = \dfrac{\dfrac{}{\Gamma,\ p : A \wedge B \ \vdash\ p : A \wedge B}\ \text{hyp}}{\Gamma,\ p : A \wedge B \ \vdash\ \mathsf{fst}\ p : A}\ \wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma,\ p : A \wedge B \ \vdash\ p : A \wedge B}\ \text{hyp}}{\Gamma,\ p : A \wedge B \ \vdash\ \mathsf{snd}\ p : B}\ \wedge\mathsf{E}^{\mathsf{L}}}{\Gamma,\ p : A \wedge B \ \vdash\ \mathsf{inl}\ (\mathsf{snd}\ p) : B \vee C}\ \vee\mathsf{I}^{\mathsf{L}}}{\Gamma,\ p : A \wedge B \ \vdash\ \langle \mathsf{fst}\ p\,, \mathsf{inl}\ (\mathsf{snd}\ p)\rangle : A \wedge (B \vee C)}\ \wedge\mathsf{I}$$

$$\text{lemma}_2 = \dfrac{\dfrac{}{\Gamma,\ q : A \wedge C \ \vdash\ q : A \wedge C}\ \text{hyp}}{\Gamma,\ q : A \wedge C \ \vdash\ \mathsf{fst}\ q : A}\ \wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma,\ q : A \wedge C \ \vdash\ q : A \wedge C}\ \text{hyp}}{\Gamma,\ q : A \wedge C \ \vdash\ \mathsf{snd}\ q : C}\ \wedge\mathsf{E}^{\mathsf{L}}}{\Gamma,\ q : A \wedge C \ \vdash\ \mathsf{inr}\ (\mathsf{snd}\ q) : B \vee C}\ \vee\mathsf{I}^{\mathsf{L}}}{\Gamma,\ q : A \wedge C \ \vdash\ \langle \mathsf{fst}\ q\,, \mathsf{inr}\ (\mathsf{snd}\ q)\rangle : A \wedge (B \vee C)}\ \wedge\mathsf{I}$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \ \vdash\ d : (A \wedge B) \vee (A \wedge C)}\ \text{hyp} \qquad \text{lemma}_1 \qquad \text{lemma}_2}{\Gamma = d : (A \wedge B) \vee (A \wedge C) \ \vdash\ \begin{array}{l}\mathsf{case}\ d\ \mathsf{of}\\ \quad \mathsf{inl}\ p \mapsto \langle \mathsf{fst}\ p\,, \mathsf{inl}\ (\mathsf{snd}\ p)\rangle \\ \quad \mathsf{inr}\ q \mapsto \langle \mathsf{fst}\ q\,, \mathsf{inr}\ (\mathsf{snd}\ q)\rangle\end{array} : A \wedge (B \vee C)}\ \vee\mathsf{E}_{p,q}}{\vdash\ \begin{array}{l}\lambda d.\ \mathsf{case}\ d\ \mathsf{of}\\ \quad \mathsf{inl}\ p \mapsto \langle \mathsf{fst}\ p\,, \mathsf{inl}\ (\mathsf{snd}\ p)\rangle \\ \quad \mathsf{inr}\ q \mapsto \langle \mathsf{fst}\ q\,, \mathsf{inr}\ (\mathsf{snd}\ q)\rangle\end{array} : (A \wedge B) \vee (A \wedge C) \to A \wedge (B \vee C)}\ \to\mathsf{I}_d$$

(n)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : A,\ f : A \to \bot \ \vdash\ f : A \to \bot}\ \text{hyp} \qquad \dfrac{}{x : A,\ f : A \to \bot \ \vdash\ x : A}\ \text{hyp}}{x : A,\ f : A \to \bot \ \vdash\ f\ x : \bot}\ \to\mathsf{E}}{x : A \ \vdash\ \lambda f.\ f\ x : (A \to \bot) \to \bot}\ \to\mathsf{I}_f}{\vdash\ \lambda x.\lambda f.\ f\ x : A \to (A \to \bot) \to \bot}\ \to\mathsf{I}_x}{\vdash\ \lambda x.\lambda f.\ f\ x : A \to \neg\neg A}\ =$$

(o)

$$\text{lemma}_1 = \dfrac{\dfrac{}{\Gamma,\ f : A \to \bot \ \vdash\ f : A \to \bot}\ \text{hyp} \qquad \dfrac{\dfrac{}{\Gamma,\ f : A \to \bot \ \vdash\ p : A \wedge B}\ \text{hyp}}{\Gamma,\ f : A \to \bot \ \vdash\ \mathsf{fst}\ p : A}\ \wedge\mathsf{E}^{\mathsf{L}}}{\Gamma,\ f : A \to \bot \ \vdash\ f\ (\mathsf{fst}\ p) : \bot}\ \to\mathsf{E}$$

$$\text{lemma}_2 = \dfrac{\dfrac{}{\Gamma,\ g : B \to \bot \ \vdash\ g : B \to \bot}\ \text{hyp} \qquad \dfrac{\dfrac{}{\Gamma,\ g : B \to \bot \ \vdash\ p : A \wedge B}\ \text{hyp}}{\Gamma,\ g : B \to \bot \ \vdash\ \mathsf{snd}\ p : B}\ \wedge\mathsf{E}^{\mathsf{L}}}{\Gamma,\ g : B \to \bot \ \vdash\ g\ (\mathsf{snd}\ p) : \bot}\ \to\mathsf{E}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \ \vdash\ d : (A \to \bot) \vee (B \to \bot)}\ \text{hyp} \qquad \text{lemma}_1 \qquad \text{lemma}_2}{\Gamma = p : A \wedge B,\ d : (A \to \bot) \vee (B \to \bot) \ \vdash\ \begin{array}{l}\mathsf{case}\ d\ \mathsf{of}\\ \quad \mathsf{inl}\ f \mapsto f\ (\mathsf{fst}\ p)\\ \quad \mathsf{inr}\ g \mapsto g\ (\mathsf{snd}\ p)\end{array} : \bot}\ \vee\mathsf{E}_{f,g}}{p : A \wedge B \ \vdash\ \begin{array}{l}\lambda d.\ \mathsf{case}\ d\ \mathsf{of}\\ \quad \mathsf{inl}\ f \mapsto f\ (\mathsf{fst}\ p)\\ \quad \mathsf{inr}\ g \mapsto g\ (\mathsf{snd}\ p)\end{array} : (A \to \bot) \vee (B \to \bot) \to \bot}\ \to\mathsf{I}_d}{\vdash\ \begin{array}{l}\lambda p.\lambda d.\ \mathsf{case}\ d\ \mathsf{of}\\ \quad \mathsf{inl}\ f \mapsto f\ (\mathsf{fst}\ p)\\ \quad \mathsf{inr}\ g \mapsto g\ (\mathsf{snd}\ p)\end{array} : A \wedge B \to (B \to \bot) \vee (B \to \bot) \to \bot}\ \to\mathsf{I}_p}{\vdash\ \begin{array}{l}\lambda p.\lambda d.\ \mathsf{case}\ d\ \mathsf{of}\\ \quad \mathsf{inl}\ f \mapsto f\ (\mathsf{fst}\ p)\\ \quad \mathsf{inr}\ g \mapsto g\ (\mathsf{snd}\ p)\end{array} : A \wedge B \to \neg(\neg A \vee \neg B)}\ =$$

(p)

$$\text{lemma}_1 = \dfrac{\dfrac{}{\Gamma,\ x : A \vdash\ p : (A \to \bot) \wedge (B \to \bot)}\ \text{hyp}}{\dfrac{\Gamma,\ x : A \vdash\ \mathsf{fst}\ p : A \to \bot}{\Gamma,\ x : A \vdash\ \mathsf{fst}\ p\ x : \bot}\ \to\mathsf{E}}\ \wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{}{\Gamma,\ x : A \vdash\ x : A}\ \text{hyp}$$

$$\text{lemma}_2 = \dfrac{\dfrac{}{\Gamma,\ y : B \vdash\ p : (A \to \bot) \wedge (B \to \bot)}\ \text{hyp}}{\dfrac{\Gamma,\ y : B \vdash\ \mathsf{snd}\ p : B \to \bot}{\Gamma,\ y : B \vdash\ \mathsf{snd}\ p\ y : \bot}\ \to\mathsf{E}}\ \wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{}{\Gamma,\ y : B \vdash\ y : B}\ \text{hyp}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash\ d : A \vee B}\ \text{hyp} \qquad \text{lemma}_1 \qquad \text{lemma}_2}{\Gamma = d : A \vee B,\ p : (A \to \bot) \wedge (B \to \bot) \vdash\ \mathsf{case}\ d\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathsf{fst}\ p\ x\ ;\ \mathsf{inr}\ y \mapsto \mathsf{snd}\ p\ y\ \} : \bot}\ \vee\mathsf{E}_{x,y}}{d : A \vee B \vdash\ \lambda p.\ \mathsf{case}\ d\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathsf{fst}\ p\ x\ ;\ \mathsf{inr}\ y \mapsto \mathsf{snd}\ p\ y\ \} : (A \to \bot) \wedge (B \to \bot) \to \bot}\ \to\mathsf{I}_p}{\vdash\ \lambda d.\lambda p.\ \mathsf{case}\ d\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathsf{fst}\ p\ x\ ;\ \mathsf{inr}\ y \mapsto \mathsf{snd}\ p\ y\ \} : A \vee B \to (A \to \bot) \wedge (B \to \bot) \to \bot}\ \to\mathsf{I}_d}{\vdash\ \lambda d.\lambda p.\ \mathsf{case}\ d\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathsf{fst}\ p\ x\ ;\ \mathsf{inr}\ y \mapsto \mathsf{snd}\ p\ y\ \} : A \vee B \to \neg(\neg A \wedge \neg B)}\ =$$

(q)

$$\text{lemma}_1 = \dfrac{\dfrac{}{\Gamma \vdash\ f : A \to B \wedge A}\ \text{hyp} \qquad \dfrac{}{\Gamma \vdash\ x : A}\ \text{hyp}}{\dfrac{\Gamma \vdash\ f\ x : B \wedge A}{\Gamma \vdash\ \mathsf{fst}\ (f\ x) : B}\ \wedge\mathsf{E}^{\mathsf{L}}}\ \to\mathsf{E}$$

$$\text{lemma}_2 = \dfrac{\dfrac{}{\Gamma \vdash\ f : A \to B \wedge A}\ \text{hyp} \qquad \dfrac{}{\Gamma \vdash\ x : A}\ \text{hyp}}{\dfrac{\Gamma \vdash\ f\ x : B \wedge A}{\Gamma \vdash\ \mathsf{snd}\ (f\ x) : A}\ \wedge\mathsf{E}^{\mathsf{R}}}\ \to\mathsf{E}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash\ g : B \to A \to C \wedge A}\ \text{hyp} \qquad \text{lemma}_1}{\Gamma \vdash\ g\ (\mathsf{fst}\ (f\ x)) : A \to C \wedge A}\ \to\mathsf{E} \qquad \text{lemma}_2}{\Gamma = f : A \to B \wedge A,\ g : B \to A \to C \wedge A,\ x : A \vdash\ g\ (\mathsf{fst}\ (f\ x))\ (\mathsf{snd}\ (f\ x)) : C \wedge A}\ \to\mathsf{E}}{f : A \to B \wedge A,\ g : B \to A \to C \wedge A \vdash\ \lambda x.\ g\ (\mathsf{fst}\ (f\ x))\ (\mathsf{snd}\ (f\ x)) : A \to C \wedge A}\ \to\mathsf{I}_x}{f : A \to B \wedge A \vdash\ \lambda g.\lambda x.\ g\ (\mathsf{fst}\ (f\ x))\ (\mathsf{snd}\ (f\ x)) : (B \to A \to C \wedge A) \to A \to C \wedge A}\ \to\mathsf{I}_g}{\vdash\ \lambda f.\lambda g.\lambda x.\ g\ (\mathsf{fst}\ (f\ x))\ (\mathsf{snd}\ (f\ x)) : (A \to B \wedge A) \to (B \to A \to C \wedge A) \to A \to C \wedge A}\ \to\mathsf{I}_f$$

9. In the following proofs, question marks indicate which proof term cannot be constructed (i.e. where the proof gets stuck).

(a)

$$\dfrac{\dfrac{\dfrac{\times}{\vdash\ ?_0 : A}}{\vdash\ \mathsf{inl}\ ?_0 : A \vee (A \to \bot)}\ \vee\mathsf{I}^{\mathsf{L}}}{\vdash\ \mathsf{inl}\ ?_0 : A \vee \neg A}\ =$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : A\ \vdash\ x : A}\ \mathsf{hyp}}{\times}}{x : A\ \vdash\ ?_1 : \bot}}{\vdash\ \lambda x.\ ?_1 : A \to \bot}\ \to\mathsf{I}_x}{\vdash\ \mathsf{inr}\ (\lambda x.\ ?_1) : A \vee (A \to \bot)}\ \vee\mathsf{I}^{\mathsf{R}}}{\vdash\ \mathsf{inr}\ (\lambda x.\ ?_1) : A \vee \neg A}\ =$$

(b)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\ \vdash\ f : (A \to \bot) \vee (B \to \bot) \to \bot}\ \mathsf{hyp} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\ x : A\ \vdash\ x : A}\ \mathsf{hyp}}{\times}}{\Gamma,\ x : A\ \vdash\ ?_0 : \bot}}{\Gamma\ \vdash\ \lambda x.\ ?_0 : A \to \bot}\ \to\mathsf{I}_x}{\Gamma\ \vdash\ \mathsf{inl}\ (\lambda x.\ ?_0) : (A \to \bot) \vee (B \to \bot)}\ \vee\mathsf{I}^{\mathsf{L}}}{\Gamma\ \vdash\ f\ (\mathsf{inl}\ (\lambda x.\ ?_0)) : \bot}\ \to\mathsf{E}}{\Gamma = f : (A \to \bot) \vee (B \to \bot) \to \bot\ \vdash\ \bot\mathsf{elim}\ (f\ (\mathsf{inl}\ (\lambda x.\ ?_0))) : A \wedge B}\ \bot\mathsf{E}}{\vdash\ \lambda f.\ \bot\mathsf{elim}\ (f\ (\mathsf{inl}\ (\lambda x.\ ?_0))) : ((A \to \bot) \vee (B \to \bot) \to \bot) \to A \wedge B}\ \to\mathsf{I}_f}{\vdash\ \lambda f.\ \bot\mathsf{elim}\ (f\ (\mathsf{inl}\ (\lambda x.\ ?_0))) : \neg(\neg A \vee \neg B) \to A \wedge B}\ =$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\ \vdash\ f : (A \to \bot) \vee (B \to \bot) \to \bot}\ \mathsf{hyp} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\ y : B\ \vdash\ y : B}\ \mathsf{hyp}}{\times}}{\Gamma,\ y : B\ \vdash\ ?_0 : \bot}}{\Gamma\ \vdash\ \lambda y.\ ?_0 : B \to \bot}\ \to\mathsf{I}_x}{\Gamma\ \vdash\ \mathsf{inr}\ (\lambda y.\ ?_0) : (A \to \bot) \vee (B \to \bot)}\ \vee\mathsf{I}^{\mathsf{R}}}{\Gamma\ \vdash\ f\ (\mathsf{inr}\ (\lambda y.\ ?_0)) : \bot}\ \to\mathsf{E}}{\Gamma = f : (A \to \bot) \vee (B \to \bot) \to \bot\ \vdash\ \bot\mathsf{elim}\ (f\ (\mathsf{inr}\ (\lambda y.\ ?_0))) : A \wedge B}\ \bot\mathsf{E}}{\vdash\ \lambda f.\ \bot\mathsf{elim}\ (f\ (\mathsf{inr}\ (\lambda y.\ ?_0))) : ((A \to \bot) \vee (B \to \bot) \to \bot) \to A \wedge B}\ \to\mathsf{I}_f}{\vdash\ \lambda f.\ \bot\mathsf{elim}\ (f\ (\mathsf{inr}\ (\lambda y.\ ?_0))) : \neg(\neg A \vee \neg B) \to A \wedge B}\ =$$

(c)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f : (A \to \bot) \wedge (B \to \bot) \to \bot}\,\text{hyp} \quad \dfrac{\dfrac{\dfrac{}{\Gamma,\ x : A \vdash x : A}\,\text{hyp}}{\Gamma \vdash \lambda x.\ ?_0 : A \to \bot}\,{\to}\mathsf{I}_x \quad \dfrac{\dfrac{}{\Gamma,\ y : B \vdash y : B}\,\text{hyp}}{\Gamma \vdash \lambda y.\ ?_1 : B \to \bot}\,{\to}\mathsf{I}_y}{\Gamma \vdash \langle \lambda x.\ ?_0 , \lambda y.\ ?_1 \rangle : (A \to \bot) \wedge (B \to \bot)}\,\wedge\mathsf{I}}{\Gamma \vdash f\ \langle \lambda x.\ ?_0 , \lambda y.\ ?_1 \rangle : \bot}\,{\to}\mathsf{E}}{\Gamma = f : (A \to \bot) \wedge (B \to \bot) \to \bot \vdash \bot\text{elim}\ (f\ \langle \lambda x.\ ?_0 , \lambda y.\ ?_1 \rangle) : A \vee B}\,\bot\mathsf{E}}{\vdash \lambda f.\ \bot\text{elim}\ (f\ \langle \lambda x.\ ?_0 , \lambda y.\ ?_1 \rangle) : ((A \to \bot) \wedge (B \to \bot) \to \bot) \to A \vee B}\,{\to}\mathsf{I}_f$$

$$\vdash \lambda f.\ \bot\text{elim}\ (f\ \langle \lambda x.\ ?_0 , \lambda y.\ ?_1 \rangle) : \neg(\neg A \wedge \neg B) \to A \vee B \quad =$$

(d)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{f : (A \to \bot) \to \bot,\ x : A \vdash x : A}\,\text{hyp}}{f : (A \to \bot) \to \bot,\ x : A \vdash\ ?_0 : \bot}}{f : (A \to \bot) \to \bot \vdash \lambda x.\ ?_0 : A \to \bot}\,{\to}\mathsf{I}_x \quad \dfrac{}{f : (A \to \bot) \to \bot}\,\text{hyp}}{f : (A \to \bot) \to \bot \vdash f\ (\lambda x.\ ?_0) : \bot}\,{\to}\mathsf{E}}{f : (A \to \bot) \to \bot \vdash \bot\text{elim}\ (f\ (\lambda x.\ ?_0)) : A}\,\bot\mathsf{E}}{\vdash \lambda f.\ \bot\text{elim}\ (f\ (\lambda x.\ ?_0)) : ((A \to \bot) \to \bot) \to A}\,{\to}\mathsf{I}_f$$

$$\vdash \lambda f.\ \bot\text{elim}\ (f\ (\lambda x.\ ?_0)) : \neg\neg A \to A \quad =$$

10. (a) Unprovable

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash d : A \vee B}\,\text{hyp} \quad \dfrac{\dfrac{}{\Gamma,\ x : A \vdash x : A}\,\text{hyp} \quad \Gamma,\ x : A \vdash\ ?_0 : B}{\Gamma,\ x : A \vdash \langle x , ?_0 \rangle : A \wedge B}\,\wedge\mathsf{I} \quad \dfrac{\Gamma,\ y : B \vdash\ ?_1 : A \quad \dfrac{}{\Gamma,\ y : B \vdash y : B}\,\text{hyp}}{\Gamma,\ y : B \vdash \langle ?_1 , y \rangle : A \wedge B}\,\wedge\mathsf{I}}{\Gamma = d : A \vee B \vdash \text{case}\ d\ \text{of}\ \{\ \text{inl}\ x \mapsto \langle x , ?_0 \rangle\ ;\ \text{inr}\ y \mapsto \langle ?_1 , y \rangle\ \} : A \wedge B}\,\vee\mathsf{E}_{x,y}}{\vdash \lambda d.\ \text{case}\ d\ \text{of}\ \{\ \text{inl}\ x \mapsto \langle x , ?_0 \rangle\ ;\ \text{inr}\ y \mapsto \langle ?_1 , y \rangle\ \} : A \vee B \to A \wedge B}\,{\to}\mathsf{I}_d$$

(b) Provable

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash g : B \to \bot}\,\text{hyp} \quad \dfrac{\dfrac{}{\Gamma \vdash f : A \to B}\,\text{hyp} \quad \dfrac{}{\Gamma \vdash x : A}\,\text{hyp}}{\Gamma \vdash f\ x : B}\,{\to}\text{E}}{\Gamma = f : A \to B,\ g : B \to \bot,\ x : A \vdash g\ (f\ x) : \bot}\,{\to}\text{E}}{f : A \to B,\ g : B \to \bot \vdash \lambda x.\ g\ (f\ x) : A \to \bot}\,{\to}\text{I}_x}{f : A \to B \vdash \lambda g.\lambda x.\ g\ (f\ x) : (B \to \bot) \to A \to \bot}\,{\to}\text{I}_g}{\vdash \lambda f.\lambda g.\lambda x.\ g\ (f\ x) : (A \to B) \to (B \to \bot) \to A \to \bot}\,{\to}\text{I}_f}{\vdash \lambda f.\lambda g.\lambda x.\ g\ (f\ x) : (A \to B) \to \neg B \to \neg A}\,=$$

(c) Unprovable

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\ y : B \vdash y : B}\,\text{hyp}}{\times}}{\Gamma,\ y : B \vdash ?_0 : \bot}}{\Gamma \vdash \lambda y.\ ?_0 : B \to \bot}\,{\to}\text{I}_y \quad \dfrac{}{\Gamma \vdash f : (B \to \bot) \to A \to \bot}\,\text{hyp}}{\Gamma \vdash f\ (\lambda y.\ ?_0) : A \to \bot}\,{\to}\text{E} \quad \dfrac{}{\Gamma \vdash x : A}\,\text{hyp}}{\Gamma \vdash f\ (\lambda y.\ ?_0)\ x : \bot}\,{\to}\text{E}}{\Gamma = f : (B \to \bot) \to A \to \bot,\ x : A \vdash \bot\text{elim}\ (f\ (\lambda y.\ ?_0)\ x) : B}\,\bot\text{E}}{f : (B \to \bot) \to A \to \bot \vdash \lambda x.\bot\text{elim}\ (f\ (\lambda y.\ ?_0)\ x) A \to B}\,{\to}\text{I}_x}{\vdash \lambda f.\lambda x.\bot\text{elim}\ (f\ (\lambda y.\ ?_0)\ x) : ((B \to \bot) \to A \to \bot) \to A \to B}\,{\to}\text{I}_f}{\vdash \lambda f.\lambda x.\bot\text{elim}\ (f\ (\lambda y.\ ?_0)\ x) : (\neg B \to \neg A) \to A \to B}\,=$$

(d) Unprovable

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{f : A \to B,\ x : A \vdash x : A}\,\mathsf{hyp}}{\times}}{f : A \to B,\ x : A \vdash\ ?_0 : \bot}\,{\to}\mathsf{I}_x}{f : A \to B \vdash\ \lambda x.\ ?_0 : A \to \bot}\,{\vee}\mathsf{I}^{\mathsf{L}}}{f : A \to B \vdash\ \mathsf{inl}\ (\lambda x.\ ?_0) : (A \to \bot) \vee B}\,{\to}\mathsf{I}_f}{\vdash\ \lambda f.\ \mathsf{inl}\ (\lambda x.\ ?_0) : (A \to B) \to (A \to \bot) \vee B}\,{=}}{\vdash\ \lambda f.\ \mathsf{inl}\ (\lambda x.\ ?_0) : (A \to B) \to \neg A \vee B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{f : A \to B,\ x : A \vdash f : A \to B}\,\mathsf{hyp}\quad\dfrac{}{f : A \to B,\ x : A \vdash x : A}\,\mathsf{hyp}}{f : A \to B,\ x : A \vdash f\ x : B}\,{\to}\mathsf{E}}{\times}}{f : A \to B,\ x : A \vdash\ ?_1 : \bot}\,{\to}\mathsf{I}_x}{f : A \to B \vdash\ \lambda x.\ ?_1 : A \to \bot}\,{\vee}\mathsf{I}^{\mathsf{L}}}{f : A \to B \vdash\ \mathsf{inl}\ (\lambda x.\ ?_1) : (A \to \bot) \vee B}\,{\to}\mathsf{I}_f}{\vdash\ \lambda f.\mathsf{inl}\ (\lambda x.\ ?_1) : (A \to B) \to (A \to \bot) \vee B}\,{=}}{\vdash\ \lambda f.\mathsf{inl}\ (\lambda x.\ ?_1) : (A \to B) \to \neg A \vee B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{f : A \to B \vdash f : A \to B}\,\mathsf{hyp}\quad\dfrac{\times}{f : A \to B \vdash\ ?_2 : A}}{f : A \to B \vdash\ f\ ?_2 : B}\,{\to}\mathsf{E}}{f : A \to B \vdash\ \mathsf{inr}\ (f\ ?_2) : (A \to \bot) \vee B}\,{\vee}\mathsf{I}^{\mathsf{R}}}{\vdash\ \lambda f.\ \mathsf{inr}\ (f\ ?_2) : (A \to B) \to (A \to \bot) \vee B}\,{\to}\mathsf{I}_f}{\vdash\ \lambda f.\ \mathsf{inr}\ (f\ ?_2) : (A \to B) \to \neg A \vee B}\,{=}$$

(e) Provable

$$\text{lemma} = \dfrac{\dfrac{}{\Gamma,\ f : A \to \bot \ \vdash\ f : A \to \bot}\ \text{hyp} \qquad \dfrac{}{\Gamma,\ f : A \to \bot \ \vdash\ x : A}\ \text{hyp}}{\dfrac{\Gamma,\ f : A \to \bot \ \vdash\ f\ x : \bot}{\Gamma,\ f : A \to \bot \ \vdash\ \bot\text{elim}\ (f\ x) : B}\ \bot\text{E}}\ \to\text{E}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \ \vdash\ d : (A \to \bot) \vee B}\ \text{hyp} \qquad \text{lemma} \qquad \dfrac{}{\Gamma,\ y : B \ \vdash\ y : B}\ \text{hyp}}{\Gamma = d : (A \to \bot) \vee B,\ x : A \ \vdash\ \text{case}\ d\ \text{of}\ \{\ \text{inl}\ f \mapsto \bot\text{elim}\ (f\ x)\ ;\ \text{inr}\ y \mapsto y\ \} : B}\ \vee\text{E}_{f,y}}{d : (A \to \bot) \vee B \ \vdash\ \lambda x.\ \text{case}\ d\ \text{of}\ \{\ \text{inl}\ f \mapsto \bot\text{elim}\ (f\ x)\ ;\ \text{inr}\ y \mapsto y\ \} : A \to B}\ \to\text{I}_x}{\vdash\ \lambda d.\lambda x.\ \text{case}\ d\ \text{of}\ \{\ \text{inl}\ f \mapsto \bot\text{elim}\ (f\ x)\ ;\ \text{inr}\ y \mapsto y\ \} : ((A \to \bot) \vee B) \to A \to B}\ \to\text{I}_d}{\vdash\ \lambda d.\lambda x.\ \text{case}\ d\ \text{of}\ \{\ \text{inl}\ f \mapsto \bot\text{elim}\ (f\ x)\ ;\ \text{inr}\ y \mapsto y\ \} : \neg A \vee B \to A \to B}\ =$$

A.3 Problems from just section 2.3.7

15. (a)

$$\dfrac{\dfrac{}{\Gamma,\ x : A \ \vdash\ x : A}\ \text{hyp} \qquad \dfrac{}{\Gamma,\ x : A \ \vdash\ \text{tt} : \top}\ \top\text{I}}{\dfrac{\Gamma,\ x : A \ \vdash\ \langle x\,, \text{tt}\rangle : A \wedge \top}{\Gamma \vdash\ \lambda x.\ \langle x\,, \text{tt}\rangle : A \to A \wedge \top}\ \to\text{I}_x}\ \wedge\text{I}$$

(b)

$$\dfrac{\dfrac{\dfrac{}{\Gamma,\ p : A \wedge \top \ \vdash\ p : A \wedge \top}\ \text{hyp}}{\Gamma,\ p : A \wedge \top \ \vdash\ \text{fst}\ p : A}\ \wedge\text{E}^{\text{L}}}{\Gamma \vdash\ \lambda p.\ \text{fst}\ p : A \wedge \top \to A}\ \to\text{I}_p$$

(c) In either possible proof, one element of the pair can be made to type check by forcing the context to contain the appropriate typing for the variable x, but this prevents the other element of the pair from type checking.

$$\dfrac{\dfrac{}{\Gamma,\ x : A \ \vdash\ x : A}\ \text{hyp} \qquad \overset{\text{X}}{\Gamma,\ x : A \ \vdash\ x : B}}{\Gamma,\ x : A \ \vdash\ \langle x\,, x\rangle : A \wedge B}\ \wedge\text{I}$$

$$\dfrac{\overset{\text{X}}{\Gamma,\ x : B \ \vdash\ x : A} \qquad \dfrac{}{\Gamma,\ x : B \ \vdash\ x : B}\ \text{hyp}}{\Gamma,\ x : B \ \vdash\ \langle x\,, x\rangle : A \wedge B}\ \wedge\text{I}$$

(d)

$$\dfrac{\dfrac{\dfrac{}{\Gamma,\ x : A \vdash x : A}\ \mathsf{hyp}}{\Gamma,\ x : A \vdash \mathsf{inl}\ x : A \vee B}\ \vee\mathsf{I}^{\mathsf{L}} \qquad \dfrac{}{\Gamma,\ x : A \vdash x : A}\ \mathsf{hyp}}{\Gamma,\ x : A \vdash \langle \mathsf{inl}\ x\,, x \rangle : (A \vee B) \wedge A}\ \wedge\mathsf{I}$$

(e) Similar to before, we can force one occurrence of x to be type checkable, but this prevents the other from also being type checkable.

$$\dfrac{\dfrac{\dfrac{}{\Gamma,\ x : B \vdash x : B}\ \mathsf{hyp}}{\Gamma,\ x : B \vdash \mathsf{inr}\ x : A \vee B}\ \vee\mathsf{I}^{\mathsf{L}} \qquad \mathsf{X}}{\Gamma,\ x : B \vdash \langle \mathsf{inr}\ x\,, x \rangle : (A \vee B) \wedge A}\ \wedge\mathsf{I}$$

$$\dfrac{\dfrac{\dfrac{\mathsf{X}}{\Gamma,\ x : A \vdash x : B}}{\Gamma,\ x : A \vdash \mathsf{inr}\ x : A \vee B}\ \vee\mathsf{I}^{\mathsf{L}} \qquad \dfrac{}{\Gamma,\ x : A \vdash x : A}\ \mathsf{hyp}}{\Gamma,\ x : A \vdash \langle \mathsf{inr}\ x\,, x \rangle : (A \vee B) \wedge A}\ \wedge\mathsf{I}$$

16. (a) No, because inl produces a disjunction, but snd requires a conjunction and so cannot take $\mathsf{inl}\ (g\ z)$ as its input.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\ g : ?_0 \to ?_1,\ x : ?_0 \vdash g : ?_0 \to ?_1}\ \mathsf{hyp} \qquad \dfrac{}{\Gamma,\ g : ?_0 \to ?_1,\ x : ?_0 \vdash x : ?_0}\ \mathsf{hyp}}{\Gamma,\ g : ?_0 \to ?_1,\ x : ?_0 \vdash g\ x : ?_1}\ \to\mathsf{E}}{\Gamma,\ g : ?_0 \to ?_1,\ x : ?_0 \vdash \mathsf{inl}\ (g\ x) : ?_1 \vee ?_2}\ \vee\mathsf{I}^{\mathsf{L}}}{\Gamma,\ g : ?_0 \to ?_1,\ x : ?_0 \vdash \mathsf{snd}\ (\mathsf{inl}\ (g\ x)) : ?_3}\ \wedge\mathsf{E}^{\mathsf{R}}$$

(b) No, because λ produces an implication, but fst requires a conjunction and so cannot take $\lambda y.y$ as its input.

$$\dfrac{\dfrac{}{\Gamma,\ y : ?_0 \ \vdash\ y : ?_0}\ \mathsf{hyp}}{\Gamma\ \vdash\ \lambda y.y : ?_0 \to ?_0}\ \to \mathsf{I}_y$$

$$\mathsf{X}$$

$$\dfrac{\Gamma\ \vdash\ \lambda y.y : ?_1 \wedge ?_2}{\Gamma\ \vdash\ \mathsf{fst}\ (\lambda y.y) : ?_1}\ \wedge \mathsf{E}^{\mathsf{L}}$$

(c) No, because it would require the type of x to be an implication, the antecedent of which is the whole implication.

$$\dfrac{\dfrac{\dfrac{}{\Gamma,\ x : ?_0 \to ?_1 \ \vdash\ x : ?_0 \to ?_1}\ \mathsf{hyp} \qquad \dfrac{}{\Gamma,\ x : ?_0 \to ?_1 \ \vdash\ x : ?_0 \to ?_1}\ \mathsf{hyp}}{\Gamma,\ x : ?_0 \to ?_1 \ \vdash\ x\ x : ?_1}\ \to \mathsf{E}\ \text{(requiring } ?_0 = ?_0 \to ?_1\text{, impossible)}}{\Gamma\ \vdash\ \lambda x.\ x\ x : (?_0 \to ?_1) \to ?_1}\ \to \mathsf{I}_x}{\Gamma\ \vdash\ \lambda x.\ x\ x : ?_0 \to ?_1}\ =$$

17. (a) $\langle x\,, y\rangle[z/x] \ = \ \langle x[z/x]\,, y[z/x]\rangle \ = \ \langle z\,, y\rangle$

(b) $\langle x\,, y\rangle[\mathsf{fst}\ p/y] \ = \ \langle x[\mathsf{fst}\ p/y]\,, y[\mathsf{fst}\ p/y]\rangle \ = \ \langle x\,, \mathsf{fst}\ p\rangle$

(c) $(\mathsf{fst}\ ((\lambda x.y)\ z))[w/x] \ = \ \mathsf{fst}\ ((\lambda x.y)\ z)[w/x] \ = \ \mathsf{fst}\ ((\lambda x.y)[w/x]\ z[w/x])$
$\mathsf{fst}\ ((\lambda x.y)\ z)$

(d) $(\mathsf{fst}\ ((\lambda x.y)\ z))[w/y] \ = \ \mathsf{fst}\ ((\lambda x.y)\ z)[w/y] \ = \ \mathsf{fst}\ ((\lambda x.y)[w/y]\ z[w/y])$
$\mathsf{fst}\ ((\lambda x.y[w/y])\ z) \ = \ \mathsf{fst}\ ((\lambda x.w)\ z)$

(e) $(\mathsf{fst}\ ((\lambda x.y)\ z))[w/z] \ = \ \mathsf{fst}\ ((\lambda x.y)\ z)[w/z] \ = \ \mathsf{fst}\ ((\lambda x.y)[w/z]\ z[w/z])$
$\mathsf{fst}\ ((\lambda x.y[w/z])\ w) \ = \ \mathsf{fst}\ ((\lambda x.y)\ w)$

18. (a) $(\lambda x.x)\ (\mathsf{fst}\ \mathcal{P}) \rightsquigarrow_\beta \mathsf{fst}\ \mathcal{P}$

(b) $(\lambda x.x)\ (\mathsf{fst}\ \langle \mathcal{A}, \mathcal{B} \rangle) \rightsquigarrow_\beta \mathsf{fst}\ \langle \mathcal{A}, \mathcal{B} \rangle \rightsquigarrow_\beta \mathcal{A}$

or

$(\lambda x.x)\ (\mathsf{fst}\ \langle \mathcal{A}, \mathcal{B} \rangle) \rightsquigarrow_\beta (\lambda x.x)\ \mathcal{A} \rightsquigarrow_\beta \mathcal{A}$

(c) $\mathsf{snd}\ ((\lambda x.\lambda y.\ \langle x, y \rangle)\ \mathcal{A}\ \mathcal{B}) \rightsquigarrow_\beta \mathsf{snd}\ ((\lambda y.\langle \mathcal{A}, y \rangle)\ \mathcal{B}) \rightsquigarrow_\beta \mathsf{snd}\ \langle \mathcal{A}, \mathcal{B} \rangle \rightsquigarrow_\beta \mathcal{B}$