Homework 1

Due 7 October

For this homework, we will introduce two new kinds of propositions. The first kind is a propositional variable, which we will represent with τ (or if we need more than one, τ_0, τ_1, τ_2, etc.). Propositional variables are stand ins for propositions, just as proof variables like x and y were stand ins for proofs. In much the same way x means "I'll give you a proof eventually, but pretend you have it now, and call it x", a propositional variable τ means "I'll give you a proposition eventually, but pretend you have it now, and call it τ". The second new kind of proposition, a quantified proposition, is analogous to a hypothetical proof with λ. Just as $\lambda x.\mathcal{B}$ means that the assumed proof x is discharged, the proposition $\forall\tau.S$ means that the assumed proposition τ is also discharged. Here are the formation, introduction, and elimination rules for the $\forall$ quantifier:

$$\frac{X \text{ prop}}{\forall\tau.X \text{ prop}}\ \forall\mathsf{F}\ \text{(if } \tau \text{ is free in } X\text{)}$$

$$\frac{\Gamma\ \vdash\ \mathcal{M} : X}{\Gamma\ \vdash\ \mathcal{M} : \forall\tau.X}\ \forall\mathsf{I}_\tau\ \text{(if } \tau \text{ is free in } X \text{ and is not used in } \Gamma\text{)}$$

$$\frac{\Gamma\ \vdash\ \mathcal{M} : \forall\tau.X}{\Gamma\ \vdash\ \mathcal{M} : X[Y/\tau]}\ \forall\mathsf{E}$$

The things in parentheses are additional requirements that have to hold in order for the inference rule to be used. So, in order to use $\forall\mathsf{F}$ to check that a quantified proposition is in fact a proposition and not gibberish, it has to be true that τ is free inside the scope X (that is to say, there is no other $\forall\tau$ somewhere in X). Similarly, to actually prove a quantified proposition is true, τ must be free, and also not used anywhere in the context Γ. Lastly, to eliminate a $\forall$, you just strip off the quantification, and replace τ with any proposition Y in the remaining scope. I should emphasize: any time you use a propositional variable like τ, you *must* eventually discharge it using $\forall\mathsf{I}$.

Let's look at some simple example proofs just to see how this works. First, let's show that $\lambda x.x$ is a proof of $\forall\tau.\tau \to \tau$, from no assumptions. We begin as always:

$$\begin{array}{c} \vdots \\ \vdash\ \lambda x.x : \forall\tau.\tau \to \tau \end{array}$$

We have no hypotheses, so we can't use an elim rule top down, nor can we just use a hypothesis to prove the goal $\forall\tau.\tau \to \tau$, so the only thing we can do is try to use $\forall\mathsf{I}$. Since X is $\tau \to \tau$, our condition that τ is free in X is satisfied, and since Γ is nothing, our condition that τ is not in Γ is also satisfied. So we can indeed apply this rule:

$$\dfrac{\begin{array}{c} \vdots \\ \vdash\ \lambda x.x : \tau \to \tau \end{array}}{\vdash\ \lambda x.x : \forall\tau.\tau \to \tau}\ \forall\mathsf{I}_\tau$$

At this point the proof is a boring old proof, so without explaining each step, we get in the end:

$$\dfrac{\dfrac{\dfrac{}{x : \tau\ \vdash\ x : \tau}\ \mathsf{hyp}}{\vdash\ \lambda x.x : \tau \to \tau}\ {\to}\mathsf{I}_x}{\vdash\ \lambda x.x : \forall\tau.\tau \to \tau}\ \forall\mathsf{I}_\tau$$

Now let's have an example of using the $\forall\mathsf{E}$ rule. Let's prove that if we have an assumption $b : \forall\tau.\tau$, we can prove $\bot$. It's quite simple:

$$\begin{array}{c} \vdots \\ b : \forall\tau.\tau\ \vdash\ \square : \bot \end{array}$$

We have no intros for $\bot$, and not hypothesis of type $\bot$, so the only thing we can try is an elim on $\forall\tau.\tau$.

$$\begin{array}{c} \dfrac{\dfrac{}{b : \forall\tau.\tau\ \vdash\ b : \forall\tau.\tau}\ \mathsf{hyp}}{b : \forall\tau.\tau\ \vdash\ b : \tau[Y/\tau]}\ \forall\mathsf{E} \\ \vdots \\ b : \forall\tau.\tau\ \vdash\ \square : \bot \end{array}$$

We have to pick a Y to substitute for τ, so that we can eventually prove $\bot$. Obviously, if we choose Y to be $\bot$ itself, then we get $\tau[\bot/\tau] = \bot$, which is our goal. So we end up with

$$\dfrac{\dfrac{}{b : \forall\tau.\tau \ \vdash \ b : \forall\tau.\tau}\ \mathsf{hyp}}{b : \forall\tau.\tau \ \vdash \ b : \bot}\ \forall\mathsf{E}$$

These two propositions — $\forall\tau.\tau$ and $\forall\tau.\tau \to \tau$ — are in fact equivalent to $\bot$ and $\top$, respectively. That is to say, $\bot \dashv\vdash \forall\tau.\tau$ and $\top \dashv\vdash \forall\tau.\tau \to \tau$, so we could in fact get rid of the $\bot$ connective and the $\top$ connective entirely, because they're now redundant.

Let's now prove something slightly more interesting, namely that $A \dashv\vdash \forall\tau.(A \to \tau) \to \tau$. In the forward direction, the proof is relatively simple:

$$\begin{array}{c}\vdots \\ x : A \ \vdash \ \square : \forall\tau.(A \to \tau) \to \tau\end{array}$$

The only thing we can try is $\forall\mathsf{I}$, and the side condition is satisfied, so we can indeed apply that rule:

$$\dfrac{\begin{array}{c}\vdots \\ x : A \ \vdash \ \square : (A \to \tau) \to \tau\end{array}}{x : A \ \vdash \ \square : \forall\tau.(A \to \tau) \to \tau}\ \forall\mathsf{I}_\tau$$

At this point, we just continue applying the old rules we've been using, and we end up with the complete proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : A, f : A \to \tau \ \vdash \ f : A \to \tau}\ \mathsf{hyp} \qquad \dfrac{}{x : A, f : A \to \tau \ \vdash \ x : A}\ \mathsf{hyp}}{x : A, f : A \to \tau \ \vdash \ f\ x : \tau}\ {\to}\mathsf{E}}{x : A \ \vdash \ \lambda f.f\ x : (A \to \tau) \to \tau}\ {\to}\mathsf{I}_f}{x : A \ \vdash \ \lambda f.f\ x : \forall\tau.(A \to \tau) \to \tau}\ \forall\mathsf{I}_\tau$$

The reverse direction requires a little more guesswork, but isn't too bad either:

$$\begin{array}{c}\vdots \\ f : \forall\tau.(A \to \tau) \to \tau \ \vdash \ \square : A\end{array}$$

The only thing we can do is elim the $\forall$. We have to pick a proposition to substitution, so we ought to look at the scope of the quantifier to see what we're going to do with it. The scope is $(A \to \tau) \to \tau$. We can continue to elim on this to yield a proof of whatever we chose for τ. So if we choose for τ to become A when doing the $\forall\mathsf{E}$, we can eventually elim down to a proof of our goal A. So that's what we choose:

$$\dfrac{\dfrac{}{f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; f : \forall\tau.(A \to \tau) \to \tau}\,\mathsf{hyp}}{f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; f : (A \to A) \to A}\,\forall\mathsf{E}$$

$$\vdots$$

$$f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; \boxed{} : A$$

Now we can apply $\to\mathsf{E}$, which introduces a new goal while also solving our previous goal:

$$\dfrac{\dfrac{\dfrac{}{f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; f : \forall\tau.(A \to \tau) \to \tau}\,\mathsf{hyp}}{f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; f : (A \to A) \to A}\,\forall\mathsf{E} \qquad \begin{array}{c}\vdots \\ f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; \boxed{} : A \to A\end{array}}{f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; \boxed{} : A}\,\to\mathsf{E}$$

But this new goal is quite easy to prove using $\to\mathsf{I}$ and hyp, so we end up with:

$$\dfrac{\dfrac{\dfrac{}{f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; f : \forall\tau.(A \to \tau) \to \tau}\,\mathsf{hyp}}{f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; f : (A \to A) \to A}\,\forall\mathsf{E} \qquad \dfrac{\dfrac{}{f : \forall\tau.(A \to \tau) \to \tau, x : A \;\vdash\; x : A}\,\mathsf{hyp}}{f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; \lambda x.x : A \to A}\,\to\mathsf{I}_x}{f : \forall\tau.(A \to \tau) \to \tau \;\vdash\; f(\lambda x.x) : A}\,\to\mathsf{E}$$

Problems on next page.

1 Problem 1

Show that with $\forall$, $\wedge$ is redundant, by proving both directions of $A \wedge B \dashv\vdash \forall \tau.(A \to B \to \tau) \to \tau$. In other words, complete these two incomplete proofs (including filling in proof terms):

$$\begin{gathered} \vdots \\ p : A \wedge B \vdash \boxed{} : \forall \tau.(A \to B \to \tau) \to \tau \end{gathered}$$

$$\begin{gathered} \vdots \\ f : \forall \tau.(A \to B \to \tau) \to \tau \vdash \boxed{} : A \wedge B \end{gathered}$$

2 Problem 2

Show that with $\forall$, $\vee$ is redundant, by proving both directions of $A \vee B \dashv\vdash \forall \tau.(A \to \tau) \to (B \to \tau) \to \tau$. In other words, complete these two incomplete proofs (including filling in proof terms):

$$\begin{gathered} \vdots \\ d : A \vee B \vdash \boxed{} : \forall \tau.(A \to \tau) \to (B \to \tau) \to \tau \end{gathered}$$

$$\begin{gathered} \vdots \\ f : \forall \tau.(A \to \tau) \to (B \to \tau) \to \tau \vdash \boxed{} : A \vee B \end{gathered}$$