Lecture Notes for LING419F
Categorial and Type-Logical Grammar

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Chapter 1

Introduction

The purpose of these lecture notes is to give a complete reference for Categorial Grammar (CG), Type-Logical Grammar (TLG), and the related topics we will be discussing in this class. But in order to fully appreciate these, it is also necessary to recall the theoretical framework that we hope to compare to: Government and Binding (GB). GB will be the source of phenomena that we need to account for, as well as the yard stick we use to measure those accounts.

At a very high level of description, we can view GB as having the following form:

$$\begin{array}{c} \text{Base Component} \\ \downarrow \\ \text{D-structure Filters} \\ \downarrow \\ \text{Transformations} \\ \downarrow \\ \text{S-structure Filters} \\ \downarrow \\ \text{Transformations} \\ \downarrow \\ \text{LF Filters} \end{array}$$

The base component and the transformations are construed generative processes that build or alter structure, while the filters are tests that structures must pass in order to proceed. Any structure that makes its way through all the filters constitutes a grammatical structure, according to the theory. The base component and transformations are, on the whole, rather simple, while the filters tend to be somewhat more complex. The Empty Category Principle, for instance, has three sub-principles, each with multiple clauses, dependent on still other definitions, and so forth.

By contrast, CG and TLG can best be viewed as just a base component, with no transformations, no filters, and no distinction between D-structure, S-structure, and LF. The generative rules that CG and TLG employ operate directly at the surface, and tend to be slightly richer than the rules found in the GB base component. Also unlike GB is an extremely fundamental dependence on proof theory and concepts originating in logic and type theory.

The structure of these notes will proceed as follows. In Chapter 2, we will discuss the necessary logical and proof theoretical preliminaries in detail. In Chapter 3, we will cover the Lambek calculus in its associative form, including the semantic component, and in Chapter 4, we will address some of its limitations by introducing a non-associative variant. These three chapters constitute the foundational core of these notes. The remaining chapters continue refining the categorial systems, pushing towards further richness and expressivity with the aim of addressing various advanced phenomena in syntactic theory.

Chapter 2

Intuitionistic Logic

Natural Deduction is a framework for logical systems that is widely used. Intuitionistic Logic is a particular logic that we will discuss using Natural Deduction. Intuitionistic Logic is not directly relevant to the syntactic topics we will be discussing later, but it does offer a language-independent introduction to the core of the proof-theoretic tools we will be using. It is, however, potentially useful for the semantic component, as that will take place in an intuitionistic setting.

The general Natural Deduction framework can be viewed as having two major components, the **propositions** that we prove things about on the one hand, and the **proofs** on the other. Propositions can further be broken down into **atomic** propositions, which have no connective and are considered "non-logical" because they represent concepts independent of the system of reasoning, and **compound** propositions, which have a **connective** together possibly with smaller propositions and other things, and are termed "logical" because they represent concepts determined by the system of reasoning. The proofs can also be broken into atomic and compound forms, with atomic proofs consisting of **axioms** and **hypotheses**, and compound proofs consisting of smaller proofs and other things combined with an **inference rule**.

2.1 Intuitionistic Propositions

The atomic propositions of Intuitionistic Logic are represented by Latin letters, typically A, B, C, etc. or P, Q, R, etc. The particular atomic propositions of a system are usually specified before doing any proofs of the system, and for this presentation we can pretend like we have all the atomic propositions we need. The connectives are more interesting. There are three binary connectives — $\wedge$, $\vee$, and $\to$ — which are written as infixes in between two propositions. These are called **conjunction**, **disjunction**, and **implication**. They are not propositions themselves, being connectives, but if, for example, we have two propositions φ and ψ, then we can combine them with these connectives to make new propositions, as in $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, and $(\varphi \to \psi)$.[1] There are also two nullary connectives — $\top$ and $\bot$, called **top** and **bottom** — which can stand alone as full fledged propositions. The intuitive meaning of these connectives, respectively, is "and", "or", "implies", "trivially true", and "false" / "absurd".

Any proposition that uses only atomic propositions together with these connectives, and does so correctly, is said to be **well-formed**.

2.1.1 Parentheses and Associativity

Conventional notation for writing compound propositions which use the binary connectives is to write parentheses around all propositions formed by these connectives, as shown above. A rather large proposition can have lots of parentheses, as in $((A \wedge B) \vee (C \to D))$. There is a convention similar to PEMDAS in mathematics, however, which allows us to eliminate some parentheses without risk of confusion. In acronymic form, it would be PNCDI, or, parentheses, negation, conjunction, disjunction, implication. The outermost parentheses around the whole proposition are obvious, as well. We would say that conjunction "binds tighter" than disjunction and implication, meaning that an unparenthesized proposition like $\varphi \wedge \psi \vee \chi$ unambiguously means $(\varphi \wedge \psi) \vee \chi$, and $\varphi \vee \psi \wedge \chi$ unambiguously means $\varphi \vee (\psi \wedge \chi)$. Similarly, both conjunction and disjunction bind tighter than implication, so that, for instance, $A \wedge B \to C$ unambiguously means $(A \wedge B) \to C$, and $A \to B \vee C$ unambiguously means $A \to (B \vee C)$. Because the symbols we are using for conjunction and disjunction have very similar visual impact, the unparenthesized forms are actually a bit hard to read, so we will always parenthesize those when different connectives are in use, to make it easier to read. For example, we will write $(\varphi \wedge \psi) \vee \chi$, keeping the parentheses in because we use both $\wedge$ and $\vee$, but we will also write $\varphi \wedge \psi \wedge \chi$, because only $\wedge$ is used. However other presentations might use different notations that make these parenthesization conventions more applicable.

Another aspect of this convention is what is often called the **associativ-**

[1] Lower case Greek letters are used to represent arbitrary propositions throughout.

ity of the connectives. When we have the option to drop parentheses around a connective, and the neighboring connective happens to the the same, the resulting expression is ambiguous, at least given what has been said so far. Thus, the proposition $\varphi \wedge \psi \wedge \chi$ could be parenthesized as either $(\varphi \wedge \psi) \wedge \chi$ or as $\varphi \wedge (\psi \wedge \chi)$. By convention, we pick the latter parenthesization as the one that can be unambiguously deparenthesized, and make it impossible to remove the parentheses on the former. Thus, if you see $\varphi \wedge \psi \wedge \chi$ it means $\varphi \wedge (\psi \wedge \chi)$, and if you mean $\varphi \wedge (\psi \wedge \chi)$, it can be written as $\varphi \wedge \psi \wedge \chi$. But if you mean $(\varphi \wedge \psi) \wedge \chi$, that is how you must write it. This applies to disjunction and implication as well. We call this right associativity, because the connectives associate with one another by letting the connectives to their right parenthesize first.

2.1.2 Subformulas

Propositions are also sometimes called **formulas**. This is especially true when talking about **subformulas**. A subformula of a proposition φ is any proposition that forms part of φ, including φ itself. For example, suppose that we consider the proposition $(A \wedge B) \vee C$. This has five subformulas: $(A \wedge B) \vee C$ itself, the two immediate subformulas $A \wedge B$ and C, which were combined with the connective $\vee$, and the two immediate subformulas of $A \wedge B$ — A and B — which were combined with the connective $\wedge$.

We can give names to the immediate subformulas of a proposition based on what the **main connective** is. In a proposition $\varphi \wedge \psi$, where the main connective is conjunction, we call φ the **left conjunct** and ψ the **right conjunct**. In $\varphi \vee \psi$, where the main connective is disjunction, we similarly call φ the **left disjunct** and ψ the **right disjunct**. And finally, in $\varphi \rightarrow \psi$, where the main connective is implication, we call φ the **antecedent** and ψ the **consequent**. The nullary connectives $\top$ and $\bot$ have no subformulas other than themselves.

When we have iterated conjunction (disjunction), it is common to describe the subformulas by a number, even tho they are all technically left or right conjuncts (disjuncts). Thus for example, in the proposition $\varphi \wedge \psi \wedge \chi$, we would normally refer to φ as the first conjunct, ψ as the second, and χ as the third, even tho technically ψ is the left conjunct of the right conjunct, and so on.

2.1.3 Exercises

1. Which of the following are well-formed propositions? Which are not, and why?

   (a) A
   (b) $A \to \bot$
   (c) $(A\wedge) \vee B$
   (d) $AB \to C \wedge D$
   (e) $\top \leftarrow A$

2. For each of the following fully parenthesized propositions, give (1) all the connectives used, (2) the main connective, (3) all atomic subformulas, and (4) all compound subformulas.

   (a) $\top$
   (b) $\bot$
   (c) A
   (d) $(A \wedge B)$
   (e) $(A \to (B \to C))$
   (f) $((A \wedge B) \vee (C \to D))$

3. What is the left conjunct in $(A \to B) \wedge C$? What is the right conjunct?

4. In the proposition $A \vee B \vee (C \to D)$, which kind of subformula (left conjunct, right disjunct, etc.) is C?

5. For each of the following partially parenthesized propositions, give the fully parenthesized form.

   (a) $A \to B \to C$
   (b) $(A \to B) \to C$
   (c) $A \wedge B \vee C$
   (d) $A \wedge B \to C \wedge D$
   (e) $A \wedge B \to C \vee D$
   (f) $(A \to B) \to C \to D$

6. Give five distinct propositions that have not been given yet in these lecture notes. For atomic propositions, use A, B, C, and D, when necessary. Be sure to use each connective at least once.

2.2 Natural Deduction

A proof in Intuitionistic Logic using the Natural Deduction framework consists of a tree of propositions connected by horizontal bars that represent the inference rules used to do the proof. For example, this is a proof of the tautology $A \to A \wedge A$:

$$\cfrac{\cfrac{\cfrac{}{A}\,x \qquad \cfrac{}{A}\,x}{A \wedge A}\,\wedge\mathsf{I}}{A \to A \wedge A}\,{\to}\mathsf{I}_x$$

The definitions of the rules can be ignored right now. Just consider the structure of the proof. It makes use of two inference rules, $\wedge\mathsf{I}$ and $\to\mathsf{I}$, together with the hypothesis A. As you can see from the structure of the proof, inference rules are draw as labeled lines. Above these lines are the **premises** of the inference, and below the line is the **conclusion** of the inference. Thus the final inference

$$\cfrac{A \wedge A}{A \to A \wedge A}\,{\to}\mathsf{I}_x$$

should be understood as saying "given that we know $A \wedge A$, we can conclude $A \to A \wedge A$ using $\to\mathsf{I}$". Disregard the little x for now. In general, an inference in a proof looks like

$$\cfrac{\varphi \qquad \psi \qquad \chi}{\rho}\,\mathsf{rule}$$

and means "given that we know φ, we know ψ, and we know χ, we can conclude ρ using rule". Some inference rules have no premises. In a very real sense, this is just a special kind of inference rule: if we can have one premise, two premises, three premises, etc. why not zero premises? But inference rules with no premises are nonetheless given a special name — they are called **axioms**.

In general, the premises themselves will be at the bottom of some proof tree, as the conclusion of some inference rule, just as $A \wedge A$ is at the bottom of the proof tree

$$\cfrac{\cfrac{}{A}\,x \qquad \cfrac{}{A}\,x}{A \wedge A}\,\wedge\mathsf{I}$$

in the above proof. In such cases, this means that these have also been proven. Any proposition that is not the conclusion of some proof tree, even a trivial one, is called a **hypothesis**, and is merely something that was assumed to be true, but may not in fact be true. A hypothesis of A would be read as meaning "assuming A holds". In some sense, a proof can have as many "invisible" hypotheses as is desired. This is because there is no requirement that hypotheses must be used as premises to an inference. We could, for instance, have a proof of $A \to A \to A$, which requires an extra hypothesis for one of the two antecedent As to dangle out in space somewhere. We could put these extra dangling hypotheses in a cloud or something like that, thereby being extremely explicit, but because we can always have whatever we want in those clouds, we leave them out entirely. In a later formulation of Natural Deduction, we will in fact make them explicit.

2.2.1 Intuitionistic Proof System

We now introduce the proof system for Intuitionistic Logic. The rules for Intuitionistic Logic consist solely of rules for **introducing** connectives into proofs, and rules for **eliminating** them from proofs, but other proof systems include other kinds of proof rules as well. We name the rules by the connective, together with an I or E for introduction and elimination, respectively. For example, $\wedge\mathsf{I}$ from earlier. Some connectives have no introduction rules, and others have no elimination rules. We will start with an intuitive connective, $\wedge$.

Introduction Rules	**Elimination Rules**
$\dfrac{X \qquad Y}{X \wedge Y}\ \wedge\mathsf{I}$	$\dfrac{X \wedge Y}{X}\ \wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{X \wedge Y}{Y}\ \wedge\mathsf{E}^{\mathsf{R}}$

The $\wedge\mathsf{I}$ rule is essentially saying that if we know X holds, and we know Y holds, then we can conclude $X \wedge Y$ holds. In a rough English equivalent, if we know "Sinclair is tall", and we know "Ivanova is short", then we can conclude "Sinclair is tall and Ivanova is short".

Conversely, $\wedge\mathsf{E}^{\mathsf{L}}$ says that if we know $X \wedge Y$ holds, then we can conclude X holds, and if $\wedge\mathsf{E}^{\mathsf{R}}$ says symmetrically that if we know $X \wedge Y$ holds, then we can conclude Y. Again, in a rough English equivalent, if we know "Sinclair is tall and Ivanova is short", then we can conclude "Sinclair is tall", or alternatively, we can conclude "Ivanova is short".

The inference rules for $\to$ are slightly simpler, but contain some new things.

Introduction Rules	**Elimination Rules**
$$\dfrac{\dfrac{\overline{X}^{\,x}}{\vdots}\quad Y}{X \to Y}\to\mathsf{I}_x$$	$$\dfrac{X \to Y \qquad X}{Y}\to\mathsf{E}$$

The elimination rule is relatively simple. If we know that $X \to Y$ holds, and we also know that X holds, then by using $\to\mathsf{E}$ we can conclude that Y holds. Roughly, if we know "if Sinclair is flying a Starfury, then Sinclair is not on the space station" and we also know "Sinclair is flying a Starfury", then we can conclude "Sinclair is not on the space station".

The introduction rule is a little more complicated. These rules are technically static — they define what a proof tree can look like — but for the purposes of explaining it, we will give a slightly dynamic explanation. The $\to\mathsf{I}$ rule is saying that if we have a proof that looks like

$$\begin{matrix}\varphi\\ \vdots\\ \psi\end{matrix}$$

which proves that ψ holds, assuming that φ holds, then we can **discharge** the hypothesis by using $\to\mathsf{I}$, producing

$$\dfrac{\begin{matrix}\overline{\varphi}^{\,x}\\ \vdots\\ \psi\end{matrix}}{\varphi \to \psi}\to\mathsf{I}_x$$

The lines above the hypothesis indicates that it has been discharged, and the x on the line and on the $\to\mathsf{I}$ let us track which hypothesis that use of $\to\mathsf{I}$ discharged. We can choose any symbol instead of x, provided it does not conflict with names elsewhere. A rough English equivalent is "Assume that Sinclair is flying a Starfury. It can then be shown that Sinclair is not on the space station. Therefore, without assuming anything, we can conclude that if Sinclair is flying a Starfury, Sinclair is not on the space station".

Disjunction has a still more complex inference rule, but its complexity is in elimination.

Introduction Rules **Elimination Rules**

$$\frac{X}{X \vee Y}\,\vee\mathsf{I}^{\mathsf{L}} \qquad \frac{Y}{X \vee Y}\,\vee\mathsf{I}^{\mathsf{R}}$$

$$\dfrac{X \vee Y \qquad \begin{matrix}\overline{X}^{\,x}\\ \vdots \\ Z\end{matrix} \qquad \begin{matrix}\overline{Y}^{\,y}\\ \vdots \\ Z\end{matrix}}{Z}\,\vee\mathsf{E}_{x,y}$$

In rough English form, the introduction rules say that if we know "Sinclair is tall" we can conclude "either Sinclair is tall or Ivanova is short", and also "either Ivanova is short or Sinclair is tall". The elimination rule is rather more complicated. It says roughly that, if we know "either Sinclair is flying a Starfury or Sinclair is on Minbar", and by assuming "Sinclair is flying a Starfury" we can show that "Sinclair is not on the space station", and also by assuming "Sinclair is on Minbar" we can show that "Sinclair is not on the space station", then we can conclude "Sinclair is not on the space station".

The rules for $\top$ and $\bot$ are vastly simpler than any of the previous rules.

Introduction Rules	**Elimination Rules**
$\dfrac{}{\top}\,\top\mathsf{I}$	none

$\top$ is trivially true, so we do not need any premises to prove it, and because it is trivially true, eliminating it would yield no new information, so we do not need elimination rules.

Introduction Rules	**Elimination Rules**
none	$\dfrac{\bot}{\varphi}\,\bot\mathsf{E}$

$\bot$ is never true because it is always false, so we do not need introduction rules, and because of this, if we ever do in fact have a proof that $\bot$ holds, we can conclude anything we like because we know we are in a purely fictional situation anyway.

The above rules suffice to prove all propositions that are true in Intuitionistic Logic. Any proof that uses only these rules and uses them correctly is said to be **valid**. A valid proof with all true/provable hypotheses (or with no hypotheses at all) is said to be **sound**.

2.2.2 Example Proofs

To get an intuition for how proofs, and proving, work in this system, we will consider the simple proposition $A \wedge B \to A \wedge B$. It should obviously be true, but we will look at a number of different proofs for it, to get a sense of how we could go about proving.

The first proof we will look at is

$$\dfrac{\dfrac{}{A \wedge B}\,p}{A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p$$

This proof is extremely short, and rather boring, but lets us introduce the art of proof construction. When we start to construct this proof, we begin in a state like so:

$$\begin{array}{c}\vdots\\ A \wedge B \to A \wedge B\end{array}$$

We know we have to end up at $A \wedge B \to A \wedge B$, but we do not yet have any idea what should go in place of the dots. By inspecting the main connective of proposition at the bottom of the dots — $\to$ — we can try to use $\to\mathsf{I}$ to "undo" the final step of the proof, to yield

$$\dfrac{\begin{array}{c}\dfrac{}{A \wedge B}\,p\\ \vdots\\ A \wedge B\end{array}}{A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p$$

We now move on to fill in the dots, and we notice that the dots conclude with $A \wedge B$, and they also assume $A \wedge B$, so we do not need any proof at all, yielding

$$\dfrac{\dfrac{}{A \wedge B}\,p}{A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p$$

There is however a second proof we can produce. Rather than using no proof at all, we could alternatively inspect the main connective of the proposition at the bottom of the dots — $\wedge$ — and try to use an introduction rule on that:

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{}{A \wedge B}\,p\\ \vdots\\ A\end{array}\qquad\begin{array}{c}\dfrac{}{A \wedge B}\,p\\ \vdots\\ B\end{array}}{A \wedge B}\,{\wedge}\mathsf{I}}{A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p$$

Notice how we made an exact copy of the dots and their hypotheses. Inspecting the proposition at the bottom of the left dots, we find that it is atomic, and has no main connective. We can not do anything do it, but perhaps by using an elimination rule on the hypothesis we can make some progress. We try $\wedge\mathsf{E}^\mathsf{L}$.

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{}{A \wedge B}\,p}{A}\,\wedge\mathsf{E}^\mathsf{L} \\ \vdots \\ A\end{array} \qquad \begin{array}{c}\dfrac{}{A \wedge B}\,p \\ \vdots \\ B\end{array}}{A \wedge B}\,\wedge\mathsf{I}}{A \wedge B \to A \wedge B}\,\to\mathsf{I}_p$$

We can now get rid of those dots entirely because the proof is again trivial. The same can be done to the dots on the right using $\wedge\mathsf{E}^\mathsf{R}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\wedge B}\,p}{A}\,\wedge\mathsf{E}^\mathsf{L} \quad \dfrac{\dfrac{}{A\wedge B}\,p}{B}\,\wedge\mathsf{E}^\mathsf{R}}{A\wedge B}\,\wedge\mathsf{I}}{A\wedge B\to A\wedge B}\,\to\mathsf{I}_p$$

In fact, there are an infinite number of proofs for this proposition. For instance, here is a third proposition that re-proves internally the proposition as an auxiliary proposition:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\wedge B}\,p}{A\wedge B\to A\wedge B}\,\to\mathsf{I}_p \quad \dfrac{}{A\wedge B}\,q}{A\wedge B}\,\to\mathsf{E}}{A\wedge B\to A\wedge B}\,\to\mathsf{I}_q$$

Obviously this business of hypothesizing $A \wedge B$, using it to eliminate $\to$, and then discharging the hypothesis to reintroduce $\to$ can continue for as many iterations as desired. When attempting to construct a proof, we would like to have good ways of narrowing our search space, to make it easier to find proofs when all we care about is the existence of one. A useful heuristic for doing this is to only ever use introduction rules at the bottom of dots, and only ever use elimination rules at the top. Proofs constructed using this heuristic are called **verifications**, and anything that is provable at all is provable via a verification.

This heuristic rules out these nasty proofs that keep introducing and eliminating $\to$, you will notice, because it would have to have used an introduction rule at the top, or an elimination rule at the bottom. The two good proofs, however, both conform to this heuristic, but in different ways. The first applies rules lazily: if it can get away with not applying a rule, it will not apply a rule. The second applies rules eagerly: if it can apply a rule, it will. Obviously, the lazy approach produces a shorter proof, so a second heuristic might be: be lazy! Do not apply rules unless you have to.

Consider the proposition $B \wedge A$. Is it possible to construct this using only the hypothesis $A \wedge B$? It ought to be. We start as expected:

$$\begin{array}{c} A \wedge B \\ \vdots \\ B \wedge A \end{array}$$

We cannot be lazy, not yet, so we have to use a rule. We can try an introduction rule. The main connective is $\wedge$, so we use $\wedge\mathsf{I}$.

$$\dfrac{\begin{array}{c} A \wedge B \\ \vdots \\ B \end{array} \qquad \begin{array}{c} A \wedge B \\ \vdots \\ A \end{array}}{B \wedge A}\ \wedge\mathsf{I}$$

The left dots now only let us apply elimination rules, because the bottom proposition is atomic, so because the main connective is $\wedge$, we use $\wedge\mathsf{E^R}$. We could try $\wedge\mathsf{E^L}$, but a little foresight shows that this will get us nowhere.

$$\dfrac{\dfrac{A \wedge B}{B}\ \wedge\mathsf{E^R} \qquad \begin{array}{c} A \wedge B \\ \vdots \\ A \end{array}}{B \wedge A}\ \wedge\mathsf{I}$$

And now similarly for the right:

$$\dfrac{\dfrac{A \wedge B}{B}\ \wedge\mathsf{E^R} \qquad \dfrac{A \wedge B}{A}\ \wedge\mathsf{E^L}}{B \wedge A}\ \wedge\mathsf{I}$$

So indeed, from just an assumption of $A \wedge B$, we can prove $B \wedge A$. The fact that we have two occurrences of $A \wedge B$ is not really that important. We can pretend they are the same, or different, depending on our needs. For instance, we could discharge both of them at once by marking them with the same symbol p, to produce a proof of $A \wedge B \to B \wedge A$, or we could discharge each independently using the symbols p and q to produce a proof of $A \wedge B \to A \wedge B \to B \wedge A$

2.2.3 Normalization and Expansion

As we saw, some proofs, the non-verifications, have extra unnecessary steps that could be removed. Typically, this involves having an elimination rule for a connective apply to the result of an introduction rule for the same connective, so that we have introduced the connective only to immediately get rid of it again. We call such unnecessary steps **detours**. We can simplify such proofs using a technique called **normalization**, which produces a kind of "simplest proof" called a **normal form**. The rules used in normalization are called β **reductions**, and so sometimes a proof will be described as β**-normal**.

We will consider conjunction first, because it is again the easiest connective to understand. Suppose we have a proof that looks like this:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{A}\\ A\end{matrix}\qquad\qquad\begin{matrix}\vdots\,\mathcal{B}\\ B\end{matrix}}{A\land B}\,\land\mathsf{I}}{A}\,\land\mathsf{E}^{\mathsf{L}}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the proofs that conclude at the premises A and B, respectively, then we have made a detour that can be eliminated. Why introduce the conjunction only to eliminate it in proving A, when we already have a proof of A, namely, $\mathcal{A}$. The simplification rule is therefore the generalization of this:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{A}\\ \varphi\end{matrix}\qquad\qquad\begin{matrix}\vdots\,\mathcal{B}\\ \psi\end{matrix}}{\varphi\land\psi}\,\land\mathsf{I}}{\varphi}\,\land\mathsf{E}^{\mathsf{L}}\qquad\rightsquigarrow_{\beta}\qquad\begin{matrix}\vdots\,\mathcal{A}\\ \varphi\end{matrix}$$

We simplify the detour by using a smaller part of the proof that already proves what we originally proved with the more complicated proof. There is of course a symmetric proof for the right:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{A}\\ \varphi\end{matrix}\qquad\qquad\begin{matrix}\vdots\,\mathcal{B}\\ \psi\end{matrix}}{\varphi\land\psi}\,\land\mathsf{I}}{\psi}\,\land\mathsf{E}^{\mathsf{R}}\qquad\rightsquigarrow_{\beta}\qquad\begin{matrix}\vdots\,\mathcal{B}\\ \psi\end{matrix}$$

Similarly, for implication detours:

$$\dfrac{\dfrac{\begin{matrix}\overline{\varphi}^{\,x}\\ \vdots\,\mathcal{B}\\ \psi\end{matrix}}{\varphi\to\psi}\,{\to}\mathsf{I}_x \qquad \begin{matrix}\vdots\,\mathcal{A}\\ \varphi\end{matrix}}{\psi}\,{\to}\mathsf{E} \qquad\rightsquigarrow_\beta\qquad \begin{matrix}\vdots\,\mathcal{A}\\ \varphi\\ \vdots\,\mathcal{B}\\ \psi\end{matrix}$$

The detour for implication here is removed by using two smaller parts of the complicated proof. Because the proof $\mathcal{B}$ has a hypothesis φ, and the proof $\mathcal{A}$ is a proof of φ, we can plug $\mathcal{A}$ into $\mathcal{B}$ as the proof for φ so that we do not have a new hypothesis overall.

For disjunction we do something very similar, but like conjunction, we have two rules:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{A}\\ \varphi\end{matrix}}{\varphi\lor\psi}\,{\lor}\mathsf{I}^{\mathsf{L}} \qquad \begin{matrix}\overline{\varphi}^{\,x}\\ \vdots\,\mathcal{C}\\ \chi\end{matrix} \qquad \begin{matrix}\overline{\psi}^{\,y}\\ \vdots\,\mathcal{C}'\\ \chi\end{matrix}}{\chi}\,{\lor}\mathsf{E}_{x,y} \qquad\rightsquigarrow_\beta\qquad \begin{matrix}\vdots\,\mathcal{A}\\ \varphi\\ \vdots\,\mathcal{C}\\ \chi\end{matrix}$$

And of course symmetrically for the right disjunct:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{B}\\ \psi\end{matrix}}{\varphi\lor\psi}\,{\lor}\mathsf{I}^{\mathsf{R}} \qquad \begin{matrix}\overline{\varphi}^{\,x}\\ \vdots\,\mathcal{C}\\ \chi\end{matrix} \qquad \begin{matrix}\overline{\psi}^{\,y}\\ \vdots\,\mathcal{C}'\\ \chi\end{matrix}}{\chi}\,{\lor}\mathsf{E}_{x,y} \qquad\rightsquigarrow_\beta\qquad \begin{matrix}\vdots\,\mathcal{B}\\ \psi\\ \vdots\,\mathcal{C}'\\ \chi\end{matrix}$$

We can ignore the normalization rules for $\top$ and $\bot$ because they are tricky to justify and offer no real interesting simplifications.

We can also construct rules that make proofs slightly more complex, by adding elimination rules followed by introduction rules. We call a process **expansion**. The rules for expansion are called η **expansions**, and so if we expand until all the connectives have been eliminated and then build them all back up again, a proof is said to be **η-long**. Again starting with conjunction:

$$\begin{matrix}\vdots\,\mathcal{P}\\ \varphi\land\psi\end{matrix} \qquad\leftrightsquigarrow_\eta\qquad \dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{P}\\ \varphi\land\psi\end{matrix}}{\varphi}\,{\land}\mathsf{E}^{\mathsf{L}} \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{P}\\ \varphi\land\psi\end{matrix}}{\psi}\,{\land}\mathsf{E}^{\mathsf{R}}}{\varphi\land\psi}\,{\land}\mathsf{I}$$

We could of course continue to do this to the conjunction at the bottom, but notice that in the middle, at least, there is no conjunction, there are only the subformulas φ and ψ. For implication, we have

$$\begin{array}{c}\vdots\,\mathcal{F}\\ \varphi\to\psi\end{array} \qquad \leftrightsquigarrow_\eta \qquad \cfrac{\cfrac{\begin{array}{c}\vdots\,\mathcal{F}\\ \varphi\to\psi\end{array} \qquad \cfrac{}{\varphi}\,x}{\psi}\,\to\mathsf{E}}{\varphi\to\psi}\,\to\mathsf{I}_x$$

And finally, for disjunction:

$$\begin{array}{c}\vdots\,\mathcal{D}\\ \varphi\vee\psi\end{array} \qquad \leftrightsquigarrow_\eta \qquad \cfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ \varphi\vee\psi\end{array} \qquad \cfrac{\cfrac{}{\varphi}\,x}{\varphi\vee\psi}\,\vee\mathsf{I}^{\mathsf{L}} \qquad \cfrac{\cfrac{}{\psi}\,y}{\varphi\vee\psi}\,\vee\mathsf{I}^{\mathsf{R}}}{\varphi\vee\psi}\,\vee\mathsf{E}_{x,y}$$

As with normalization, we will ignore the $\top$ and $\bot$ expansion rules.

In some sense, both normalization rules and expansion rules are bidirectional, working in either direction, but expansion rules are more obviously so, and therefore I used a two-headed arrow for these rules to emphasize this bidirectionality.

2.2.4 Exercises

7. Give both a verification proof and a non-verification proof for each of the following propositions. Do not use the same detour twice. Overall, which kind of proof was easier to find, and why?

    (a) $A \to A$

    (b) $A \wedge A \to A$

    (c) $A \vee B \to B \vee A$

8. The following propositions can be proved in Intuitionistic Logic. Give any kind of proof at all, verification or not, for each of them. Note that $\neg\varphi$ is shorthand for $\varphi \to \bot$.

    (a) $A \to B \to A$

    (b) $A \vee A \to A$

    (c) $A \wedge B \to A \vee B$

(d) $A \to (A \to B) \to B$

(e) $(A \to B \to C) \to B \to A \to C$

(f) $(A \wedge B \to C) \to A \to B \to C$

(g) $(A \to B \to C) \to A \wedge B \to C$

(h) $(A \to B) \to (A \to C) \to A \to B \wedge C$

(i) $(A \to B) \to (C \to D) \to A \wedge C \to B \wedge D$

(j) $(A \to B) \to (B \to C) \to A \to C$

(k) $(A \to B \to C) \to (A \to B) \to A \to C$

(l) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

(m) $(A \wedge B) \vee (A \wedge C) \to A \wedge (B \vee C)$

(n) $A \to \neg\neg A$

(o) $A \wedge B \to \neg(\neg A \vee \neg B)$

(p) $A \vee B \to \neg(\neg A \wedge \neg B)$

(q) $(A \to B \wedge A) \to (B \to A \to C \wedge A) \to A \to C \wedge A$

9. The following propositions cannot be proved in Intuitionistic Logic without using assumptions. Why not? Give as much of a verification as possible, up to wherever you get stuck.

   (a) $A \vee \neg A$

   (b) $\neg(\neg A \vee \neg B) \to A \wedge B$

   (c) $\neg(\neg A \wedge \neg B) \to A \vee B$

   (d) $\neg\neg A \to A$

10. Determine whether or not the following propositions can be proved without using assumptions in Intuitionistic Logic. If so, give a proof. If not, give a verification that cannot be finished.

   (a) $A \vee B \to A \wedge B$

   (b) $(A \to B) \to \neg B \to \neg A$

   (c) $(\neg B \to \neg A) \to A \to B$

   (d) $(A \to B) \to \neg A \vee B$

   (e) $(\neg A \vee B) \to A \to B$

11. Normalize the following proofs. Show each step.

(a)
$$\dfrac{\dfrac{\dfrac{}{A}\,x}{A\to A}\to\mathsf{I}_x \qquad \dfrac{\overset{\vdots\,\mathcal{P}}{A\wedge B}}{A}\wedge\mathsf{E}^{\mathsf{L}}}{A}\to\mathsf{E}$$

(b)
$$\dfrac{\dfrac{\dfrac{}{A}\,x}{A\to A}\to\mathsf{I}_x \qquad \dfrac{\dfrac{\overset{\vdots\,\mathcal{A}}{A} \qquad \overset{\vdots\,\mathcal{B}}{B}}{A\wedge B}\wedge\mathsf{I}}{A}\wedge\mathsf{E}^{\mathsf{L}}}{A}\to\mathsf{E}$$

(c)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A}\,x \qquad \dfrac{}{B}\,y}{A\wedge B}\wedge\mathsf{I}}{B\to A\wedge B}\to\mathsf{I}_y}{A\to B\to A\wedge B}\to\mathsf{I}_x \qquad \overset{\vdots\,\mathcal{A}}{A}}{B\to A\wedge B}\to\mathsf{E} \qquad \overset{\vdots\,\mathcal{B}}{B}}{A\wedge B}\to\mathsf{E}}{B}\wedge\mathsf{E}^{\mathsf{R}}$$

12. Are there any interesting relationships between verifications and the normalization and expansion rules?

13. Is there anything special about $\wedge\mathsf{E}^{\mathsf{L}}$, $\wedge\mathsf{E}^{\mathsf{R}}$, $\to\mathsf{E}$, $\top\mathsf{I}$, $\vee\mathsf{E}$, and $\bot\mathsf{E}$ that set them apart from the other inference rules, especially in the context of constructing verifications?

2.3 Contexts and Proof Terms

Proofs in Intuitionistic Logic as presented here can be reified into things that we can talk about using meta-proofs. When we reify them, what we are doing is assigning a linear, rather than tree-like, notation to proofs called the λ Calculus. In doing this, we also introduce contexts which list all of the hypotheses we are using explicitly, rather than leaving them invisible. By doing this, it becomes possible to state proof rules as local rules that look only at the bottom-most portions of a proof, rather than the whole proof, as with the $\to\mathsf{I}$ and $\vee\mathsf{E}$ rules.

2.3.1 Contexts

A **context** is a collection of (distinct) names like x paired with a proposition, as in $x : A$. It represents a collection of hypotheses for a proof. In this setting, symbols like x are called **variables**, and propositions are called **types**. An example of a context (sometimes called a typing context) is

$$f : B \to C,\ g : A \to B,\ x : A$$

Contexts are referred to using capital Greek letters such as Γ or Δ.

The order if elements in a context is inconsequential, and so contexts will often be shuffled around when it is useful. This often goes together with using a Greek symbol to abbreviate part of a context, as in

$$\Gamma,\ x : A$$

Because of the distinctness of names, this x is understood as not occurring in Γ.

2.3.2 Proof Terms

A **proof term**, or just **term**, is a linear representation of a proof. Rather than give a translation from normal proofs into proof terms, I will just sketch the general manner of the translation. Essentially, a proof that looks like

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{A}\\ A\end{array}\qquad\begin{array}{c}\vdots\,\mathcal{B}\\ B\end{array}\qquad\begin{array}{c}\vdots\,\mathcal{C}\\ C\end{array}}{D}\,\mathsf{rule}$$

gets translated into a proof term that looks like

$$\mathsf{rule}(\mathcal{A}, \mathcal{B}, \mathcal{C})$$

In general, using the name of the inference rule is hard to read, so instead, a different notation is used. For instance, as we will see shortly, rather than writing $\wedge\mathsf{I}(\mathcal{A}, \mathcal{B})$ we just write $\langle\mathcal{A}\,, \mathcal{B}\rangle$. In this setting, the rule names are often called **constructors**.

The variant of Natural Deduction that uses contexts and proof terms builds up such terms in the course of the proof.

2.3.3 Terms in Context

We can now put together proof terms and typing contexts into a single judgment of the form

$$\Gamma \;\vdash\; \mathcal{M} : \varphi$$

which should be understood as saying that under the hypotheses in Γ, the proof term $\mathcal{M}$ proves φ. Alternatively, we can say that in context Γ, the term $\mathcal{M}$ has the **type** φ.

The symbol $\vdash$ is called the **turnstile**.

2.3.4 Intuitionistic Proof System

The full collection of inference rules for the term-in-context style is as follows.

Introduction Rules

$$\frac{\Gamma \vdash \mathcal{X} : X \qquad \Gamma \vdash \mathcal{Y} : Y}{\Gamma \vdash \langle \mathcal{X}, \mathcal{Y} \rangle : X \wedge Y} \wedge\mathsf{I}$$

$$\frac{\Gamma,\ x : X \vdash \mathcal{Y} : Y}{\Gamma \vdash \lambda x.\mathcal{Y} : X \to Y} \to\mathsf{I}_x$$

$$\frac{\Gamma \vdash \mathcal{X} : X}{\Gamma \vdash \mathsf{inl}\ \mathcal{X} : X \vee Y} \vee\mathsf{I}^{\mathsf{L}}$$

$$\frac{\Gamma \vdash \mathcal{Y} : Y}{\Gamma \vdash \mathsf{inr}\ \mathcal{Y} : X \vee Y} \vee\mathsf{I}^{\mathsf{R}}$$

$$\frac{}{\Gamma \vdash \mathsf{tt} : \top} \top\mathsf{I}$$

—

Elimination Rules

$$\frac{\Gamma \vdash \mathcal{P} : X \wedge Y}{\Gamma \vdash \mathsf{fst}\ \mathcal{P} : X} \wedge\mathsf{E}^{\mathsf{L}} \qquad \frac{\Gamma \vdash \mathcal{P} : X \wedge Y}{\Gamma \vdash \mathsf{snd}\ \mathcal{P} : Y} \wedge\mathsf{E}^{\mathsf{R}}$$

$$\frac{\Gamma \vdash \mathcal{F} : X \to Y \qquad \Gamma \vdash \mathcal{X} : X}{\Gamma \vdash \mathcal{F}\mathcal{X} : Y} \to\mathsf{E}$$

$$\frac{\Gamma \vdash \mathcal{D} : X \vee Y \qquad \Gamma,\ x : X \vdash \mathcal{Z} : Z \qquad \Gamma,\ y : Y \vdash \mathcal{Z}' : Z}{\Gamma \vdash \mathsf{case}\ \mathcal{D}\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{Z}\ ;\ \mathsf{inr}\ y \mapsto \mathcal{Z}'\ \} : Z} \vee\mathsf{E}_{x,y}$$

—

$$\frac{\Gamma \vdash \mathcal{L} : \bot}{\Gamma \vdash \bot\mathsf{elim}\ \mathcal{L} : Z} \bot\mathsf{E}$$

Note that fst , snd , and $\bot$elim are not separable from what they combine with, which is different from the $\mathcal{F}$ in $\mathcal{F}\mathcal{A}$. Also note that in an expression like $\mathcal{F}\mathcal{A}\mathcal{B}$, what is meant is $(\mathcal{F}\mathcal{A})\,\mathcal{B}$, so that this adjacency notation is left-associative.

Lastly, we have an axiom that sits somewhat outside the normal inference rules, being more a part of Natural Deduction than Intuitionistic Logic:

$$\frac{}{\Gamma,\ x : X \ \vdash\ x : X}\ \mathsf{hyp}$$

In the context of the λ Calculus, a proof of the form $\langle \mathcal{A} , \mathcal{B} \rangle$ is called a **pair**, with fst and snd being the first and second **projections**, respectively. A proof of the form $\lambda x.\mathcal{B}$ is called a λ **abstraction**, the sub-proof $\mathcal{B}$ is called the **body** of the abstraction, and $\mathcal{F}\mathcal{A}$ is called a **function application** or just an **application**, where $\mathcal{F}$ is the **function** and $\mathcal{A}$ is the **argument**. Proofs of the form inl $\mathcal{A}$ and inr $\mathcal{B}$ are called the left and right **injections**, while proofs of the form $\mathsf{case}\ \mathcal{D}\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{C}\ ;\ \mathsf{inr}\ y \mapsto \mathcal{C}'\ \}$ are called **case expressions** (hence the constructor).[2]

Similarly, we have related terminology on the type/proposition side: $\varphi \wedge \psi$ is called a **pair type** (often written $\varphi \times \psi$ in such contexts), $\varphi \to \psi$ is called a **function type**, and $\varphi \vee \psi$ is called a **sum type**. The antecedent and succedent of an implication-as-function are called the **argument type** and **return type**, respectively.

2.3.5 Examples Proofs and Term Construction

We can now redo some familiar proofs using proof terms. Recall the various proofs for $A \wedge A \to A \wedge A$. The simplest was rather trivial.

$$\frac{\dfrac{}{p : A \wedge B \ \vdash\ p : A \wedge B}\ \mathsf{hyp}}{\vdash\ \lambda p.p : A \wedge B \to A \wedge B}\ {\to}\mathsf{I}_p$$

We also saw a slightly more complex proof that was η-long:

$$\frac{\dfrac{\dfrac{\dfrac{}{p : A \wedge B \ \vdash\ p : A \wedge B}\ \mathsf{hyp}}{p : A \wedge B \ \vdash\ \mathsf{fst}\ p : A}\ \wedge\mathsf{E^L} \qquad \dfrac{\dfrac{}{p : A \wedge B \ \vdash\ p : A \wedge B}\ \mathsf{hyp}}{p : A \wedge B \ \vdash\ \mathsf{snd}\ p : B}\ \wedge\mathsf{E^R}}{p : A \wedge B \ \vdash\ \langle \mathsf{fst}\ p , \mathsf{snd}\ p \rangle : A \wedge B}\ \wedge\mathsf{I}}{\vdash\ \lambda p.\langle \mathsf{fst}\ p , \mathsf{snd}\ p \rangle : A \wedge B \to A \wedge B}\ {\to}\mathsf{I}_p$$

There was the whole class of proofs with detours, such as

[2] Case expressions can in fact be generalized to all propositions/types, but in generally this is avoided for readability. The case expression for conjunction/pairs is

$$\mathsf{case}_\times\ \mathcal{P}\ \mathsf{of}\ \{\ \langle x , y \rangle\ \mapsto \mathcal{C}\ \}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{q : A \wedge B,\ p : A \wedge B \ \vdash\ p : A \wedge B}\,\mathsf{hyp}}{q : A \wedge B \ \vdash\ \lambda p.p : A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p \qquad \dfrac{}{q : A \wedge B \ \vdash\ q : A \wedge B}\,\mathsf{hyp}}{q : A \wedge B \ \vdash\ (\lambda p.p)\, q : A \wedge B}\,{\to}\mathsf{E}}{\vdash\ \lambda q.\ (\lambda p.p)\, q : A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_q$$

We have also seen a hypothetical proof that shows that $B \wedge A$ holds under the assumption that $A \wedge B$ holds:

$$\dfrac{\dfrac{\dfrac{}{p : A \wedge B \ \vdash\ p : A \wedge B}\,\mathsf{hyp}}{p : A \wedge B \ \vdash\ \mathsf{snd}\ p : B}\,\wedge\mathsf{E}^\mathsf{R} \qquad \dfrac{\dfrac{}{p : A \wedge B \ \vdash\ p : A \wedge B}\,\mathsf{hyp}}{p : A \wedge B \ \vdash\ \mathsf{fst}\ p : A}\,\wedge\mathsf{E}^\mathsf{L}}{p : A \wedge B \ \vdash\ \langle \mathsf{snd}\ p\,, \mathsf{fst}\ p \rangle : B \wedge A}\,\wedge\mathsf{I}$$

Finding proofs in a terms-in-context system like this is exactly like finding proofs in the simpler form of Natural Deduction. The primary difference is that in constructing a proof, you have to also construct a proof term. The easiest way to do this is in a bidirectional fashion, first finding a proof for the proposition, and then by filling in the proof terms starting at the hyp steps and working down. For example, using the η-long proof as a target, we can begin proving like so:

$$\begin{array}{c} \vdots \\ \vdash\ \boxed{} : A \wedge B \to A \wedge B \end{array}$$

We leave a grey space between the turnstile and the typing colon so we know that we have to add the proof term later. Our first move now, using our heuristic for building proofs, is to use an introduction rule on $\to$:

$$\dfrac{\begin{array}{c} \vdots \\ p : A \wedge B \ \vdash\ \boxed{} : A \wedge B \end{array}}{\vdash\ \boxed{} : A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p$$

We could now use the hyp rule and complete this proof, but we are trying to construct the η-long proof, so instead we use another introduction rule on $\wedge$:

$$\dfrac{\dfrac{\begin{array}{c} \vdots \\ p : A \wedge B \ \vdash\ \boxed{} : A \end{array} \qquad \begin{array}{c} \vdots \\ p : A \wedge B \ \vdash\ \boxed{} : B \end{array}}{p : A \wedge B \ \vdash\ \boxed{} : A \wedge B}\,\wedge\mathsf{I}}{\vdash\ \boxed{} : A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p$$

We now cannot use any more introduction rules, so we switch to elimination rules:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ p : A \wedge B \vdash \boxed{} : A \wedge B\end{array}}{p : A \wedge B \vdash \boxed{} : A}\wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{\begin{array}{c}\vdots\\ p : A \wedge B \vdash \boxed{} : A \wedge B\end{array}}{p : A \wedge B \vdash \boxed{} : B}\wedge\mathsf{E}^{\mathsf{R}}}{p : A \wedge B \vdash \boxed{} : A \wedge B}\wedge\mathsf{I}}{\vdash \boxed{} : A \wedge B \to A \wedge B}\to\mathsf{I}_p$$

And now we finish up with hyp:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p : A \wedge B \vdash \boxed{} : A \wedge B}\mathsf{hyp}}{p : A \wedge B \vdash \boxed{} : A}\wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{\dfrac{}{p : A \wedge B \vdash \boxed{} : A \wedge B}\mathsf{hyp}}{p : A \wedge B \vdash \boxed{} : B}\wedge\mathsf{E}^{\mathsf{R}}}{p : A \wedge B \vdash \boxed{} : A \wedge B}\wedge\mathsf{I}}{\vdash \boxed{} : A \wedge B \to A \wedge B}\to\mathsf{I}_p$$

Having found a proof, we just need to construct the proof terms. Starting at hyp, we fill in the variable names like the hyp rule says:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p : A \wedge B \vdash p : A \wedge B}\mathsf{hyp}}{p : A \wedge B \vdash \boxed{} : A}\wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{\dfrac{}{p : A \wedge B \vdash p : A \wedge B}\mathsf{hyp}}{p : A \wedge B \vdash \boxed{} : B}\wedge\mathsf{E}^{\mathsf{R}}}{p : A \wedge B \vdash \boxed{} : A \wedge B}\wedge\mathsf{I}}{\vdash \boxed{} : A \wedge B \to A \wedge B}\to\mathsf{I}_p$$

The conjunction elimination rules now say we add either fst or snd :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p : A \wedge B \vdash p : A \wedge B}\mathsf{hyp}}{p : A \wedge B \vdash \mathsf{fst}\ p : A}\wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{\dfrac{}{p : A \wedge B \vdash p : A \wedge B}\mathsf{hyp}}{p : A \wedge B \vdash \mathsf{snd}\ p : B}\wedge\mathsf{E}^{\mathsf{R}}}{p : A \wedge B \vdash \boxed{} : A \wedge B}\wedge\mathsf{I}}{\vdash \boxed{} : A \wedge B \to A \wedge B}\to\mathsf{I}_p$$

The conjunction introduction rule says we wrap these in angle brackets:

$$\dfrac{\dfrac{\dfrac{}{p : A \wedge B \ \vdash\ p : A \wedge B}\,\mathsf{hyp}}{p : A \wedge B \ \vdash\ \mathsf{fst}\ p : A}\,\wedge\mathsf{E^L} \qquad \dfrac{\dfrac{}{p : A \wedge B \ \vdash\ p : A \wedge B}\,\mathsf{hyp}}{p : A \wedge B \ \vdash\ \mathsf{snd}\ p : B}\,\wedge\mathsf{E^R}}{\dfrac{p : A \wedge B \ \vdash\ \langle \mathsf{fst}\ p\, , \mathsf{snd}\ p \rangle : A \wedge B}{\vdash\ \boxed{} : A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p}\,\wedge\mathsf{I}$$

And finally, implication introduction says to put a λ on the front:

$$\dfrac{\dfrac{\dfrac{}{p : A \wedge B \ \vdash\ p : A \wedge B}\,\mathsf{hyp}}{p : A \wedge B \ \vdash\ \mathsf{fst}\ p : A}\,\wedge\mathsf{E^L} \qquad \dfrac{\dfrac{}{p : A \wedge B \ \vdash\ p : A \wedge B}\,\mathsf{hyp}}{p : A \wedge B \ \vdash\ \mathsf{snd}\ p : B}\,\wedge\mathsf{E^R}}{\dfrac{p : A \wedge B \ \vdash\ \langle \mathsf{fst}\ p\, , \mathsf{snd}\ p \rangle : A \wedge B}{\vdash\ \lambda p.\langle \mathsf{fst}\ p\, , \mathsf{snd}\ p \rangle : A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p}\,\wedge\mathsf{I}$$

Unsurprisingly, this matches the previous version of the η-long proof with terms.

We can also work backwards from a proof term to its proof. Suppose we want to show that $\lambda p.\langle \mathsf{fst}\ p\, , \mathsf{snd}\ p \rangle$ does indeed have type $A \wedge B \to A \wedge B$. One way to do this is to use the term to guide the typing proof:

$$\begin{array}{c} \vdots \\ \vdash\ \lambda p.\langle \mathsf{fst}\ p\, , \mathsf{snd}\ p \rangle : A \wedge B \to A \wedge B \end{array}$$

Because the proof term is a λ, we know that we have to have gotten it by using implication introduction. Fortunately, the type has $\to$ as its main connective, which can indeed be introduced via implication introduction.

$$\dfrac{\begin{array}{c} \vdots \\ p : A \wedge B \ \vdash\ \langle \mathsf{fst}\ p\, , \mathsf{snd}\ p \rangle : A \wedge B \end{array}}{\vdash\ \lambda p.\langle \mathsf{fst}\ p\, , \mathsf{snd}\ p \rangle : A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p$$

Now, since the proof term is headed by a pair, we know we have to introduce it using conjunction introduction, which the type supports.

$$\dfrac{\dfrac{\begin{array}{c} \vdots \\ p : A \wedge B \ \vdash\ \mathsf{fst}\ p : A \end{array} \qquad \begin{array}{c} \vdots \\ p : A \wedge B \ \vdash\ \mathsf{snd}\ p : B \end{array}}{p : A \wedge B \ \vdash\ \langle \mathsf{fst}\ p\, , \mathsf{snd}\ p \rangle : A \wedge B}}{\vdash\ \lambda p.\langle \mathsf{fst}\ p\, , \mathsf{snd}\ p \rangle : A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p$$

The fst and snd constructors now tell us we must apply conjunction elimination rules.

$$\dfrac{\dfrac{\dfrac{\vdots}{p : A \wedge B \;\vdash\; p : A \wedge B}}{p : A \wedge B \;\vdash\; \mathsf{fst}\ p : A}\,\wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{\dfrac{\vdots}{p : A \wedge B \;\vdash\; p : A \wedge B}}{p : A \wedge B \;\vdash\; \mathsf{snd}\ p : B}\,\wedge\mathsf{E}^{\mathsf{R}}}{\dfrac{p : A \wedge B \;\vdash\; \langle \mathsf{fst}\ p \,, \mathsf{snd}\ p \rangle : A \wedge B}{\vdash\; \lambda p.\langle \mathsf{fst}\ p \,, \mathsf{snd}\ p \rangle : A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p}$$

And now we can lastly use hyp to finish the proof.

$$\dfrac{\dfrac{\dfrac{}{p : A \wedge B \;\vdash\; p : A \wedge B}\,\mathsf{hyp}}{p : A \wedge B \;\vdash\; \mathsf{fst}\ p : A}\,\wedge\mathsf{E}^{\mathsf{L}} \qquad \dfrac{\dfrac{}{p : A \wedge B \;\vdash\; p : A \wedge B}\,\mathsf{hyp}}{p : A \wedge B \;\vdash\; \mathsf{snd}\ p : B}\,\wedge\mathsf{E}^{\mathsf{R}}}{\dfrac{p : A \wedge B \;\vdash\; \langle \mathsf{fst}\ p \,, \mathsf{snd}\ p \rangle : A \wedge B}{\vdash\; \lambda p.\langle \mathsf{fst}\ p \,, \mathsf{snd}\ p \rangle : A \wedge B \to A \wedge B}\,{\to}\mathsf{I}_p}$$

This kind of process, where we have both a term and a type and we want to check if the term does in fact have that type, is unsurprisingly called type checking, and is the bulk of what these lecture notes will deal with in syntax.

The last way we can use proof terms is to determine what type the term could have. Again we work bottom up, but we leave the type space blank.

$$\dfrac{\vdots}{\vdash\; \lambda p.\langle \mathsf{fst}\ p \,, \mathsf{snd}\ p \rangle : \boxed{}}$$

Again, the proof term starts with a λ, so we must use implication introduction:

$$\dfrac{\dfrac{\vdots}{p : \boxed{} \;\vdash\; \langle \mathsf{fst}\ p \,, \mathsf{snd}\ p \rangle : \boxed{}}}{\vdash\; \lambda p.\langle \mathsf{fst}\ p \,, \mathsf{snd}\ p \rangle : \boxed{}}\,{\to}\mathsf{I}_p$$

And now conjunction introduction:

$$\dfrac{\dfrac{\dfrac{\vdots}{p : \boxed{} \;\vdash\; \mathsf{fst}\ p : \boxed{}} \qquad \dfrac{\vdots}{p : \boxed{} \;\vdash\; \mathsf{snd}\ p : \boxed{}}}{p : \boxed{} \;\vdash\; \langle \mathsf{fst}\ p \,, \mathsf{snd}\ p \rangle : \boxed{}}\,\wedge\mathsf{I}}{\vdash\; \lambda p.\langle \mathsf{fst}\ p \,, \mathsf{snd}\ p \rangle : \boxed{}}\,{\to}\mathsf{I}_p$$

Then conjunction eliminations:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ p : \boxed{} \ \vdash\ p : \boxed{}\end{array}}{p : \boxed{} \ \vdash\ \mathsf{fst}\ p : \boxed{}}\,\wedge\mathsf{E^L} \qquad \dfrac{\begin{array}{c}\vdots\\ p : \boxed{} \ \vdash\ p : \boxed{}\end{array}}{p : \boxed{} \ \vdash\ \mathsf{snd}\ p : \boxed{}}\,\wedge\mathsf{E^R}}{p : \boxed{} \ \vdash\ \langle \mathsf{fst}\ p\,, \mathsf{snd}\ p\rangle : \boxed{}}\,\wedge\mathsf{I}}{\vdash\ \lambda p.\langle \mathsf{fst}\ p\,, \mathsf{snd}\ p\rangle : \boxed{}}\,\to\mathsf{I}_p$$

And we finish up with hyp:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p : \boxed{} \ \vdash\ p : \boxed{}}\,\mathsf{hyp}}{p : \boxed{} \ \vdash\ \mathsf{fst}\ p : \boxed{}}\,\wedge\mathsf{E^L} \qquad \dfrac{\dfrac{}{p : \boxed{} \ \vdash\ p : \boxed{}}\,\mathsf{hyp}}{p : \boxed{} \ \vdash\ \mathsf{snd}\ p : \boxed{}}\,\wedge\mathsf{E^R}}{p : \boxed{} \ \vdash\ \langle \mathsf{fst}\ p\,, \mathsf{snd}\ p\rangle : \boxed{}}\,\wedge\mathsf{I}}{\vdash\ \lambda p.\langle \mathsf{fst}\ p\,, \mathsf{snd}\ p\rangle : \boxed{}}\,\to\mathsf{I}_p$$

Now all we have to do is fill in the types. Unlike the case of filling in the terms, we have to look a little bit up and down the proof to see what kind of type we need. Starting at the left hyp, we see that it is the premise to a conjunction elimination rule, so whatever type we assign to p must be a conjunction. The same is true of the right hyp. So we can pick the general conjunction form $\varphi \wedge \psi$ and assign that to the type of p throughout the whole proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p : \varphi\wedge\psi \ \vdash\ p : \varphi\wedge\psi}\,\mathsf{hyp}}{p : \varphi\wedge\psi \ \vdash\ \mathsf{fst}\ p : \boxed{}}\,\wedge\mathsf{E^L} \qquad \dfrac{\dfrac{}{p : \varphi\wedge\psi \ \vdash\ p : \varphi\wedge\psi}\,\mathsf{hyp}}{p : \varphi\wedge\psi \ \vdash\ \mathsf{snd}\ p : \boxed{}}\,\wedge\mathsf{E^R}}{p : \varphi\wedge\psi \ \vdash\ \langle \mathsf{fst}\ p\,, \mathsf{snd}\ p\rangle : \boxed{}}\,\wedge\mathsf{I}}{\vdash\ \lambda p.\langle \mathsf{fst}\ p\,, \mathsf{snd}\ p\rangle : \boxed{}}\,\to\mathsf{I}_p$$

Next, we look at the term $\mathsf{fst}\ p$ and we notice that it is the conclusion of a conjunction elimination rule, so we choose the appropriate conjunct. Similarly for $\mathsf{snd}\ p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p : \varphi\wedge\psi \ \vdash\ p : \varphi\wedge\psi}\,\mathsf{hyp}}{p : \varphi\wedge\psi \ \vdash\ \mathsf{fst}\ p : \varphi}\,\wedge\mathsf{E^L} \qquad \dfrac{\dfrac{}{p : \varphi\wedge\psi \ \vdash\ p : \varphi\wedge\psi}\,\mathsf{hyp}}{p : \varphi\wedge\psi \ \vdash\ \mathsf{snd}\ p : \psi}\,\wedge\mathsf{E^R}}{p : \varphi\wedge\psi \ \vdash\ \langle \mathsf{fst}\ p\,, \mathsf{snd}\ p\rangle : \boxed{}}\,\wedge\mathsf{I}}{\vdash\ \lambda p.\langle \mathsf{fst}\ p\,, \mathsf{snd}\ p\rangle : \boxed{}}\,\to\mathsf{I}_p$$

We proceed in this way until we end up at the expected complete proof. Note that I wrote φ and ψ not A and B. The reason is that we do not have enough information to say exactly what φ and ψ must be: they could be A and B, or they could be some big hairy formulas, but the proof term itself does not give us enough information to decide.

Often, the type information flows very far in both directions. Consider, for instance, the following proof with empty space for types:

$$\dfrac{\dfrac{\dfrac{}{x : \square,\ y : \square \ \vdash\ x : \square}\ \mathsf{hyp}}{y : \square \ \vdash\ \lambda x.x : \square}\ {\to}\mathsf{I}_x \qquad \dfrac{}{y : \square \ \vdash\ y : \square}\ \mathsf{hyp}}{y : \square \ \vdash\ (\lambda x.x)\, y : \square}\ {\to}\mathsf{E}$$

The type of x and y are pretty much free for us to choose. But once we choose one, it turns out to force a decision for the other. Let's choose y to be of the generic type φ:

$$\dfrac{\dfrac{\dfrac{}{x : \square,\ y : \varphi \ \vdash\ x : \square}\ \mathsf{hyp}}{y : \varphi \ \vdash\ \lambda x.x : \square}\ {\to}\mathsf{I}_x \qquad \dfrac{}{y : \varphi \ \vdash\ y : \varphi}\ \mathsf{hyp}}{y : \varphi \ \vdash\ (\lambda x.x)\, y : \square}\ {\to}\mathsf{E}$$

One we have done this, we now know that the term $\lambda x.x$ must have type $\varphi \to \psi$, for some ψ, in order for it to be applied to y, which lets us fill in a lot more of the proof's empty types:

$$\dfrac{\dfrac{\dfrac{}{x : \varphi\ \ y : \varphi \ \vdash\ x : \psi}\ \mathsf{hyp}}{y : \varphi \ \vdash\ \lambda x.x : \varphi \to \psi}\ {\to}\mathsf{I}_x \qquad \dfrac{}{y : \varphi \ \vdash\ y : \varphi}\ \mathsf{hyp}}{y : \varphi \ \vdash\ (\lambda x.x)\, y : \psi}\ {\to}\mathsf{E}$$

Having filled in all the empty types, we now have to check that the hypotheses are correct. The hypothesis for y is of course correct, but we notice that the hypothesis for x has a mismatch: on the left of the turnstile, we have $x : \varphi$, but on the right we have $x : \psi$. In order to fix this, we just need to replace ψ with φ throughout the proof:

$$\dfrac{\dfrac{\dfrac{}{x : \varphi\ \ y : \varphi \ \vdash\ x : \varphi}\ \mathsf{hyp}}{y : \varphi \ \vdash\ \lambda x.x : \varphi \to \varphi}\ {\to}\mathsf{I}_x \qquad \dfrac{}{y : \varphi \ \vdash\ y : \varphi}\ \mathsf{hyp}}{y : \varphi \ \vdash\ (\lambda x.x)\, y : \varphi}\ {\to}\mathsf{E}$$

Giving it one last look over, we find nothing wrong, and we have given a full proof for the term $\lambda x.x\, y$. This kind of proof, where we discover the type of some term, is called a **type synthesis** proof.

2.3.6 Normalization and Expansion

Normalization and expansion rules carry over straight-forwardly in the obvious fashion. However, because we now have terms, we can state the normalization rules for terms rather then having to give the whole proof. The terms allow us to reconstruct the proof, if we wanted. The normalization rules on terms are

$$
\begin{array}{rcl}
\mathsf{fst}\ \langle \mathcal{X}\,, \mathcal{Y} \rangle & \leadsto_\beta & \mathcal{X} \\
\mathsf{snd}\ \langle \mathcal{X}\,, \mathcal{Y} \rangle & \leadsto_\beta & \mathcal{Y} \\
(\lambda x.\mathcal{Y})\, \mathcal{X} & \leadsto_\beta & \mathcal{Y}[\mathcal{X}/x] \\
\mathsf{case}\ (\mathsf{inl}\ \mathcal{X})\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{Z}\ ;\ \mathsf{inr}\ y \mapsto \mathcal{Z}'\ \} & \leadsto_\beta & \mathcal{Z}[\mathcal{X}/x] \\
\mathsf{case}\ (\mathsf{inr}\ \mathcal{Y})\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{Z}\ ;\ \mathsf{inr}\ y \mapsto \mathcal{Z}'\ \} & \leadsto_\beta & \mathcal{Z}'[\mathcal{Y}/y]
\end{array}
$$

The normalization rules make many of the names for the eliminations more sensible. For instance, fst clearly projects out the first element of a pair such as $\langle \mathcal{A}\,, \mathcal{B} \rangle$.

Substitution as in $\mathcal{M}[\mathcal{S}/s]$ is defined as

$$x[\mathcal{S}/s] = \mathcal{A} \quad \textit{(when } s = x\textit{)}$$

$$x[\mathcal{S}/s] = x \quad \textit{(when } s \neq x\textit{)}$$

$$\langle \mathcal{X} , \mathcal{Y} \rangle[\mathcal{S}/s] = \langle \mathcal{X}[\mathcal{S}/s] , \mathcal{Y}[\mathcal{S}/s] \rangle$$

$$(\mathsf{fst}\ \mathcal{P})[\mathcal{S}/s] = \mathsf{fst}\ (\mathcal{P}[\mathcal{S}/s])$$

$$(\mathsf{snd}\ \mathcal{P})[\mathcal{S}/s] = \mathsf{snd}\ (\mathcal{P}[\mathcal{S}/s])$$

$$(\lambda x.\mathcal{Y})[\mathcal{S}/s] = \lambda x.\mathcal{Y} \quad \textit{(when } s = x\textit{)}$$

$$(\lambda x.\mathcal{Y})[\mathcal{S}/s] = \lambda x.\mathcal{Y}[\mathcal{S}/s] \quad \textit{(when } s \neq x\textit{)}$$

$$(\mathcal{F}\,\mathcal{X})[\mathcal{S}/s] = \mathcal{F}[\mathcal{S}/s]\ \mathcal{X}[\mathcal{S}/s]$$

$$(\mathsf{inl}\ \mathcal{X})[\mathcal{S}/s] = \mathsf{inl}\ (\mathcal{X}[\mathcal{S}/s])$$

$$(\mathsf{inr}\ \mathcal{Y})[\mathcal{S}/s] = \mathsf{inr}\ (\mathcal{Y}[\mathcal{S}/s])$$

$$(\mathsf{case}\ \mathcal{D}\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{Z}\ ;\ \mathsf{inr}\ y \mapsto \mathcal{Z}'\ \})[\mathcal{S}/s] = \mathsf{case}\ \mathcal{D}[\mathcal{S}/s]\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{Z}\ ;\ \mathsf{inr}\ y \mapsto \mathcal{Z}'[\mathcal{S}/s]\ \}$$
$$\textit{(when } s = x \textit{ and } s \neq y\textit{)}$$

$$(\mathsf{case}\ \mathcal{D}\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{Z}\ ;\ \mathsf{inr}\ y \mapsto \mathcal{Z}'\ \})[\mathcal{S}/s] = \mathsf{case}\ \mathcal{D}[\mathcal{S}/s]\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{Z}[\mathcal{S}/s]\ ;\ \mathsf{inr}\ y \mapsto \mathcal{Z}'\ \}$$
$$\textit{(when } s \neq x \textit{ and } s = y\textit{)}$$

$$(\mathsf{case}\ \mathcal{D}\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{Z}\ ;\ \mathsf{inr}\ y \mapsto \mathcal{Z}'\ \})[\mathcal{S}/s] = \mathsf{case}\ \mathcal{D}[\mathcal{S}/s]\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{Z}[\mathcal{S}/s]\ ;\ \mathsf{inr}\ y \mapsto \mathcal{Z}'[\mathcal{S}/s]\ \}$$
$$\textit{(when } s \neq x \textit{ and } s \neq y\textit{)}$$

This is an explicit, structural definition of the substitution we were able to carry out before via labeled hypotheses.

2.3.7 Exercises

14. Re-do the proofs in 2.2.4 using proof terms.

15. For each of the following terms, check that it does in fact have the given type. If it does, give the proof. If it does not, give a proof that goes as far as possible, and explain why no further progress can be made.

(a) $\lambda x.\langle x\,, \mathsf{tt}\rangle : A \to A \land \top$

(b) $\lambda p.\mathsf{fst}\ p : A \land \top \to A$

(c) $\langle x\,, x\rangle : A \land B$

(d) $\langle \mathsf{inl}\ x\,, x\rangle : (A \lor B) \land A$

(e) $\langle \mathsf{inr}\ x\,, x\rangle : (A \lor B) \land A$

16. For each of the following terms, is it possible to construct a proof that corresponds to the proof term? Why or why not? Give a proof with terms, with as much progress as possible, and showing the synthesized type if there is one.

    (a) $\mathsf{snd}\ (\mathsf{inl}\ (g\ z))$

    (b) $\mathsf{fst}\ (\lambda y.y)$

    (c) $\lambda x.\, x\, x$

17. For each of the following proof terms, carry out the indicated substitution, showing each step.

    (a) $\langle x\,, y\rangle[z/x]$

    (b) $\langle x\,, y\rangle[\mathsf{fst}\ p/y]$

    (c) $(\mathsf{fst}\ ((\lambda x.y)\ z))[w/x]$

    (d) $(\mathsf{fst}\ ((\lambda x.y)\ z))[w/y]$

    (e) $(\mathsf{fst}\ ((\lambda x.y)\ z))[w/z]$

18. Normalize the following proof terms.

    (a) $(\lambda x.x)\,(\mathsf{fst}\ \mathcal{P})$

    (b) $(\lambda x.x)\,(\mathsf{fst}\ \langle \mathcal{A}\,, \mathcal{B}\rangle)$

    (c) $\mathsf{snd}\ ((\lambda x.\lambda y.\langle x\,, y\rangle)\ \mathcal{A}\ \mathcal{B})$

2.4 Sequent Calculus

The Sequent Calculus is another proof framework, like Natural Deduction with contexts, but with some subtle differences. The biggest difference is that there are no introduction and elimination rules. In their place we have **left rules** and **right rules**, which operate on the left and right sides of sequents. A **sequent** is a context together with a term and a proposition, written $\Gamma \Longrightarrow \mathcal{M} : \varphi$. In pure logic settings, the terms will often be omitted (as will variables), using just propositions, in which case a sequent looks like $\Gamma \Longrightarrow \varphi$. Terms are normal terms as before.

2.4.1 Intuitionistic Proof System

The Sequent Calculus rules for Intuitionistic Logic are a bit peculiar, so we will go through them one at a time. The right rules are similar to their Natural Deduction introduction counterparts, but the left rules are especially peculiar, and do not look like elimination rules at all. As always, starting with conjunction, and giving all rules with proof terms:

$$\frac{\Gamma \implies \mathcal{X} : X \qquad \Gamma \implies \mathcal{Y} : Y}{\Gamma \implies \langle \mathcal{X}, \mathcal{Y} \rangle : X \wedge Y} \wedge\mathsf{R}$$

This is essentially identical to conjunction introduction, except we have swapped the turnstile for a double arrow. The left rules are a bit unusual however.

$$\frac{\Gamma,\ p : X \wedge Y,\ x : X \implies \mathcal{Z} : Z}{\Gamma,\ p : X \wedge Y \implies \mathcal{Z}[\mathsf{fst}\ p/x] : \chi} \wedge\mathsf{L}^{\mathsf{L}}\mathrm{x}$$

This inference rule says that if we can give a proof $\mathcal{Z}$ of Z using Γ and $x : X$, then we must be able to give a similar proof using $p : X \wedge Y$ instead of x, by using the first component of p in place of x. So for example, we can construct the following incomplete proof of A:

$$\frac{p : A \wedge B,\ x : A \implies x : A}{p : A \wedge B \implies \mathsf{fst}\ p : A} \wedge\mathsf{L}^{\mathsf{L}}\mathrm{y}$$

Notice how the left rule takes the place of an elimination rule. There is of course a symmetric rule for the right conjunct:

$$\frac{\Gamma,\ p : X \wedge Y,\ y : Y \implies \mathcal{Z} : Z}{\Gamma,\ p : X \wedge Y \implies \mathcal{Z}[\mathsf{snd}\ p/y] : Z} \wedge\mathsf{L}^{\mathsf{R}}\mathrm{y}$$

The right rule for implication is familiar:

$$\frac{\Gamma,\ x : X \implies \mathcal{Y} : Y}{\Gamma \implies \lambda x.\mathcal{Y} : X \to Y} \to\mathsf{R}\mathrm{x}$$

But the left rule is rather peculiar:

$$\frac{\Gamma,\ f : X \to Y \implies \mathcal{X} : X \qquad \Gamma,\ f : X \to Y,\ y : Y \implies \mathcal{Z} : Z}{\Gamma,\ f : X \to Y \implies \mathcal{Z}[f\mathcal{X}/y] : Z} \to \mathsf{Ly}$$

Essentially, it is saying that we can introduce a function application in place of y, introducing the function variable f in the process, provided we already have the argument of the function.

The right rules for disjunction are also familiar:

$$\frac{\Gamma \implies \mathcal{X} : X}{\Gamma \implies \mathsf{inl}\ \mathcal{X} : X \lor Y} \lor\mathsf{R}^{\mathsf{L}}$$

$$\frac{\Gamma \implies \mathcal{Y} : Y}{\Gamma \implies \mathsf{inl}\ \mathcal{Y} : X \lor Y} \lor\mathsf{R}^{\mathsf{R}}$$

The left rule for disjunction, however, is rather peculiar, much like the left rule for implication:

$$\frac{\Gamma,\ d : X \lor Y,\ x : X \implies \mathcal{Z} : Z \qquad \Gamma,\ d : X \lor Y,\ y : Y \implies \mathcal{Z}' : Z}{\Gamma,\ d : X \lor Y \implies \mathsf{case}\ d\ \mathsf{of}\ \{\ \mathsf{inl}\ x \mapsto \mathcal{Z}\ ;\ \mathsf{inr}\ y \mapsto \mathcal{Z}'\ \} : Z} \lor\mathsf{Lxy}$$

Finally, for $\top$ we have only a right rule, just as we have only an introduction rule in Natural Deduction:

$$\frac{}{\Gamma \implies \mathsf{tt} : \top} \top\mathsf{R}$$

And for $\bot$ we have only a left rule, just as we have only an elimination rule in Natural Deduction:

$$\frac{}{\Gamma,\ x : \bot \implies \bot\mathsf{elim}\ x : X} \bot\mathsf{L}$$

Like Natural Deduction, the Sequent Calculus also has a more general, logic-independent proof rule for variables:

$$\frac{}{\Gamma,\ x : X \implies x : X} \mathsf{ID}$$

As with Natural Deduction before, the contexts in the Sequent Calculus have no intrinsic ordering or structure to them, they are just an unstructured collection of hypotheses.

Using just these inference rules, we can construct all and only the verifications proof terms from Natural Deduction. The left rules are essentially equivalent to eliminations operating on the top of a Natural Deduction proof, and right rules are essentially equivalent to introductions operating on the bottom. Because these rules yield verification terms, there are no detours in the terms, and so no reductions can apply. If we want to be able to reconstruct all Natural Deduction proofs, not just the verifications, we need to introduce another rule called **cut** for adding detours:

$$\frac{\Gamma \implies \mathcal{X} : X \qquad \Gamma,\ x : X \implies \mathcal{Y} : Y}{\Gamma \implies \mathcal{Y}[\mathcal{X}/x] : Y}\ \mathsf{CUTx}$$

Because contexts have distinct variables, all new variables introduced to the context by inference rules must be new, and not appear in the rest of the context.

2.4.2 Example Proofs and Term Construction

We will consider now η-long proofs of $A \wedge B \to A \wedge B$, in the Sequent Calculus without proof terms, and then with. The process of finding these proofs works best bottom-up, starting with the conclusion we are trying to show and working towards axioms. On the other hand, constructing the proof terms works best top-down, starting with the axioms, much like in Natural Deduction. We will step through this process using a termless proof. We start with

$$\begin{array}{c} \vdots \\ \implies A \wedge B \to A \wedge B \end{array}$$

As the main connective of the proposition is $\to$, and it is on the right of $\implies$, we can use $\to\mathsf{R}$:

$$\frac{\begin{array}{c} \vdots \\ A \wedge B \implies A \wedge B \end{array}}{\implies A \wedge B \to A \wedge B}\ \to\mathsf{R}$$

At this point, we have two options. We can use a left rule for conjunction, or a right rule. We will try to use a right rule:

$$\frac{\dfrac{\begin{array}{c} \vdots \\ A \wedge B \implies A \end{array} \qquad \begin{array}{c} \vdots \\ A \wedge B \implies B \end{array}}{A \wedge B \implies A \wedge B}\ \wedge\mathsf{R}}{\implies A \wedge B \to A \wedge B}\ \to\mathsf{R}$$

At this point, neither of the sub-problems of the proof can proceed by using right rules, so we must use a left rule. We will use $\wedge\mathsf{L}^{\mathsf{L}}$ for the left sub-problem, and $\wedge\mathsf{L}^{\mathsf{R}}$ for the right.

$$\dfrac{\dfrac{\dfrac{\vdots}{A \wedge B,\ A \implies A}}{A \wedge B \implies A}\,\wedge\mathsf{L}^{\mathsf{L}} \qquad \dfrac{\dfrac{\vdots}{A \wedge B,\ B \implies B}}{A \wedge B \implies B}\,\wedge\mathsf{L}^{\mathsf{R}}}{\dfrac{A \wedge B \implies A \wedge B}{\implies A \wedge B \rightarrow A \wedge B}\,{\rightarrow}\mathsf{R}}\,\wedge\mathsf{R}$$

We can finish up with ID:

$$\dfrac{\dfrac{\dfrac{}{A \wedge B,\ A \implies A}\,\mathsf{ID}}{A \wedge B \implies A}\,\wedge\mathsf{L}^{\mathsf{L}} \qquad \dfrac{\dfrac{}{A \wedge B,\ B \implies B}\,\mathsf{ID}}{A \wedge B \implies B}\,\wedge\mathsf{L}^{\mathsf{R}}}{\dfrac{A \wedge B \implies A \wedge B}{\implies A \wedge B \rightarrow A \wedge B}\,{\rightarrow}\mathsf{R}}\,\wedge\mathsf{R}$$

If we want to now annotate proof terms. The ID rules are simple enough, we just choose variable names, the left conjunction rules introduce a new name for the pair plus the projections, the right conjunction rule just adds the pair syntax, and the implication right rule introduces a λ abstraction.

$$\dfrac{\dfrac{\dfrac{}{p : A \wedge B,\ x : A \implies x : A}\,\mathsf{ID}}{p : A \wedge B \implies \mathsf{fst}\ p : A}\,\wedge\mathsf{L}^{\mathsf{L}}\mathrm{x} \qquad \dfrac{\dfrac{}{p : A \wedge B,\ y : B \implies y : B}\,\mathsf{ID}}{p : A \wedge B \implies \mathsf{snd}\ p : B}\,\wedge\mathsf{L}^{\mathsf{R}}\mathrm{y}}{\dfrac{p : A \wedge B \implies \langle \mathsf{fst}\ p\,, \mathsf{snd}\ p\rangle : A \wedge B}{\implies \lambda p.\langle \mathsf{fst}\ p\,, \mathsf{snd}\ p\rangle : A \wedge B \rightarrow A \wedge B}\,{\rightarrow}\mathsf{R}\mathrm{p}}\,\wedge\mathsf{R}$$

The more direct proof from earlier is of course similarly simple:

$$\dfrac{\dfrac{}{p : A \wedge B \implies p : A \wedge B}\,\mathsf{ID}}{\implies \lambda p.p : A \wedge B \rightarrow A \wedge B}\,{\rightarrow}\mathsf{R}\mathrm{p}$$

A more interesting situation arises when we need to use cut, as in the case of the first of the "circuitous" proofs from before, which constructed the proof of $A \wedge B \rightarrow A \wedge B$ just to discharge its hypothesis. The Sequent Calculus equivalent is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{q : A \wedge B,\ p : A \wedge B \implies p : A \wedge B}\,\mathsf{ID}}{q : A \wedge B \implies \lambda p.p : A \wedge B \rightarrow A \wedge B}\,{\rightarrow}\mathsf{R}\mathrm{p} \qquad \dfrac{\dfrac{}{q : A \wedge B \implies q : A \wedge B}\,\mathsf{ID} \quad \dfrac{}{q : A \wedge B,\ r : A \wedge B \implies r : A \wedge B}\,\mathsf{ID}}{q : A \wedge B,\ f : A \wedge B \rightarrow A \wedge B \implies f\,q : A \wedge B}\,{\rightarrow}\mathsf{L}\mathrm{r}}{q : A \wedge B \implies (\lambda p.p)\,q : A \wedge B}\,\mathsf{CUT}\mathrm{f}}{\implies \lambda q.(\lambda p.p)\,q : A \wedge B}\,{\rightarrow}\mathsf{R}\mathrm{q}$$

Constructing this proof from scratch is actually somewhat tricky to do, and requires a bit of insight. However, if we had the proof term, constructing the proof can be quite simple, because the proof term can only be constructed in certain ways, and thus limits the rules we could have used.

2.4.3 Normalization and Expansion

Normalization in the Sequent Calculus amounts to something called **cut elimination**. Just as in Natural Deduction we could prove all provable things without detours, in the Sequent Calculus we can prove all provable things without using cut. However, defining cut elimination is tedious. Similar things can be said about expansion rules. Fortunately, normalization and expansion can be performed on proof terms using the same term rules as in Natural Deduction, so we can Sequent Calculus versions in these lecture notes.

2.4.4 Exercises

19. Re-do the proofs of the relevant propositions in 2.2.4 in the Sequent Calculus, using proof terms.

20. The following propositions can be proved in Intuitionistic Logic. Give Sequent Calculus proofs for each of them. Give the proof with terms.

    (a) $A \to A \wedge A$

    (b) $(\top \to A) \to A$

    (c) $A \vee B \to B \vee A$

    (d) $(A \vee B \to C) \to A \to C$

    (e) $(A \to C) \to (B \to C) \to A \vee B \to C$

    (f) $A \wedge (B \to C) \to B \to A \wedge C$

    (g) $(A \vee \neg A) \to \neg\neg A \to A$

    (h) $(A \to (B \to C) \to C) \to ((A \to C) \to C) \to (B \to C) \to C$

    (i) $(A \wedge ((B \to C) \to A) \to A) \to (A \wedge (B \to A) \to A) \to A \wedge (C \to A) \to A$

21. Is there any nice structural relationship between Natural Deduction elimination rules and Sequent Calculus left rules?

22. Could we use a different rule for left conjunction instead of the one given? If so, provide it, and construct a proof of $A \wedge B \to B \wedge A$.