Chapter 3

Associative Lambek Calculus

The general shape of the CG theory of natural language syntax is that natural language can be modeled as language-specific lexicons, which consider of axioms in the proof theory being used, together with the background proof rules of the proof theory. In a way, this is similar to the base component of GB, which has lexical entries together with general base component principles. CG is somewhat more rich in what can exist in the lexical entries, however. There, a word-as-axiom can have any type that the particular logic allows. Because the theory is essentially just a logic plus some non-logical axioms, the details are in the logic, and the first of these that we will explore is the Associative Lambek Calculus.

The Associative Lambek Calculus is very similar to Intuitionistic Logic. Its main points of divergence are the connectives involved, and the structure of collections of assumptions. We will have to be extremely careful when dealing with these collection assumptions, because their structure now becomes relevant to the rules. Before, with Intuitionistic Logic, the assumptions were just a set of named propositions, where order did not matter. They were just some sort of amorphous cloud floating in the background. In the Associative Lambek Calculus, they are a list, where order matters.

3.1 Associative Lambek Calculus Types

Propositions in the Associative Lambek Calculus are called **types**, or in a more linguistic mode, **categories** (hence "categorial" in Categorial Grammar). As with Intuitionistic Logic propositions, the Associative Lambek Calculus consists of a number of atomic types A, B, C, etc. together with connectives. There are three connectives in the Associative Lambek Calculus: $\otimes$, $\backslash$, and $/$, called **product**, **left implication**, and **right implication**. Occasionally, the term division is used instead of implication. These connectives are read as “times”, “under”, and “over”, respectively. So, for example, $A \otimes B$ is A times B, $A \backslash B$ is A under B, and B/A is B over A.

The subformulas of the product can be referred to with the same names as for conjunction, because they are almost identical. Similarly, left and right implication are very similar to implication, and so we can use the same names for its subformulas as for normal implication: in both $A \backslash B$ and B/A, the subformula A is the antecedent/argument type, and B is the consequent/return type. The slash tilts towards the argument type.

3.2 Natural Deduction

Because there are only three connectives, two of which are very similar, the proof rules for the Associative Lambek Calculus are relatively simple. However, it must be said before hand that in this logic, assumptions are strictly additive. That is to say, different subproofs have strictly disjoint assumptions. This is not obvious in the Natural Deduction presentation, however.

For product we have the rules

Introduction Rules	**Elimination Rules**
$\dfrac{X \qquad Y}{X \otimes Y}\ \otimes\mathsf{I}$	$\dfrac{X \otimes Y}{X \qquad Y}\ \otimes\mathsf{E}$

The $\otimes\mathsf{E}$ rule has a funny Natural Deduction presentation: it has two simultaneous conclusions! This is because the logic in question is “linear” in the sense that we cannot throw away or ignore any sub-propositions.

For the implications, we have two very similar, but left-to-right mirrored rules.

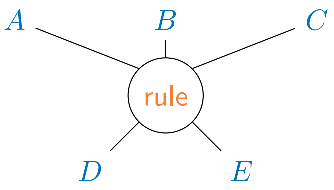
Introduction Rules	**Elimination Rules**
$\begin{array}{c} \overline{X}^{\,x}\ \ldots \\ \vdots \\ \dfrac{Y}{X\backslash Y}\,\backslash\mathsf{I}_x \end{array}$	$\dfrac{X \qquad X\backslash Y}{Y}\,\backslash\mathsf{E}$
$\begin{array}{c} \ldots\ \overline{X}^{\,x} \\ \vdots \\ \dfrac{Y}{Y/X}\,/\mathsf{I}_x \end{array}$	$\dfrac{Y/X \qquad X}{Y}\,/\mathsf{E}$

Again it must be stressed that order of parts of a proof is important. The assumption x in the $\backslash\mathsf{I}$ rule must be the left-most inference (hypothesis or not) over the use of that rule, and similarly x must be the right-most over the use of the $/\mathsf{I}$ rule — any inference rule, whether it is a hypothesis or not, that prevents the hypotheses from being on the left (or right) edge also prevents the $\backslash\mathsf{I}$ and $/\mathsf{I}$ rules from being applicable.

These inference rules, or specifically, the $\otimes\mathsf{E}$ rule, make the usual proof style with inference lines somewhat hard to follow. Because of this, the Associative Lambek Calculus lets us use what are called **proof nets**. Instead of writing lines with inference rule labels like this:

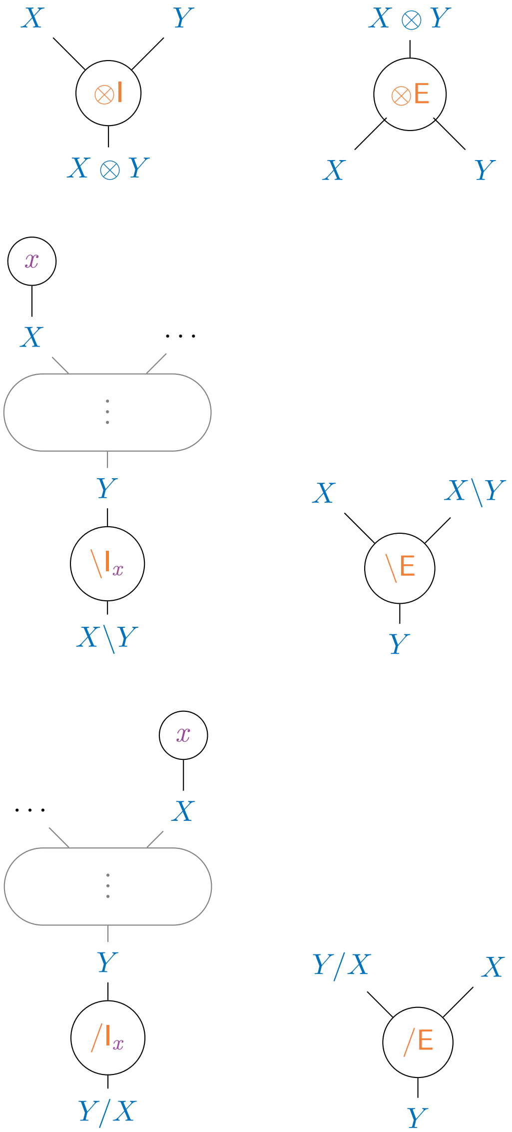
$$\dfrac{A \qquad B \qquad C}{D \qquad E}\,\mathsf{rule}$$

we can instead write a little circle with the rulename in it, and with lines connecting up to premises and down to conclusions, like this:



This is a slightly bulkier notation, but vastly easier to use for writing proofs. So our inference rules in this notation are

Introduction Rules **Elimination Rules**



Some caution must be taken here: because the ⊗E rule has multiple conclusions, and therefore because any given partial proof could have multiple conclusions, it is absolutely important that the \I and /I rules apply to subproofs that have exactly one conclusion. In other words, the $\vdots$ portion of the above inferences rules must have exactly one conclusion: Y.

3.2.1 Examples

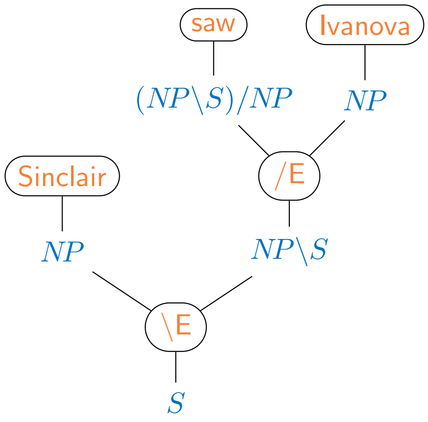
To give examples, it will be useful to have some lexical typing axioms, so that the examples can be of a syntactic nature, rather than a purely logical nature.

$$\dfrac{}{NP}\,\textsf{Sinclair} \qquad \dfrac{}{NP}\,\textsf{Ivanova} \qquad \dfrac{}{(NP\backslash S)/NP}\,\textsf{saw}$$

We can now give some simple examples. A sentence such as *Sinclair saw Ivanova* is taken to be grammatical just in case it has a proof of the type S (for sentence, of course). That is to say, the appropriate lexical axioms are used, in the appropriate order, together with other inference rules, to prove S. This is typically said more concisely as *Sinclair saw Ivanova* : S, which foreshadows how proof terms will be used, albeit imprecisely. This sentence therefore corresponds to the proof

$$\dfrac{\dfrac{}{NP}\,\textsf{Sinclair} \quad \dfrac{\dfrac{}{(NP\backslash S)/NP}\,\textsf{saw} \quad \dfrac{}{NP}\,\textsf{Ivanova}}{NP\backslash S}\,/\textsf{E}}{S}\,\backslash\textsf{E}$$

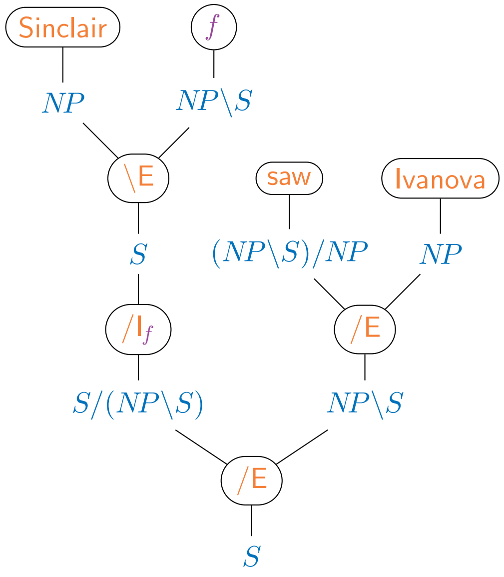
Or, using the proof net notation:



There are plenty of other proofs of this, however. For example, here is one where we introduce a right implication detour:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{NP}\,\textsf{Sinclair} \quad \dfrac{}{NP\backslash S}\,f}{S}\,\backslash\textsf{E}}{S/(NP\backslash S)}\,/\textsf{I}_f \quad \dfrac{\dfrac{}{(NP\backslash S)/NP}\,\textsf{saw} \quad \dfrac{}{NP}\,\textsf{Ivanova}}{NP\backslash S}\,/\textsf{E}}{S}\,/\textsf{E}$$

Using a proof net:



Notice that the hypothesis f does not prevent this from being a proof that corresponds to the sentence *Sinclair saw Ivanova*. In this style of proofs — with tree proofs or proof nets — this is basically a stipulation, but when we have contexts and proof terms, it will become something that we actually capture in proof terms.

3.2.2 Exercises

23. Assume the above axioms for Sinclair, Ivanova, and saw. Add to these the following axioms:

$$\frac{}{(NP\backslash S)/NP}\ \textsf{flew} \qquad \frac{}{NP\backslash S}\ \textsf{spoke} \qquad \frac{}{NP/N}\ \textsf{the} \qquad \frac{}{NP/N}\ \textsf{a}$$

$$\frac{}{N}\ \textsf{Starfury} \qquad \frac{}{N}\ \textsf{mission} \qquad \frac{}{(NP\backslash NP)/NP}\ \textsf{and} \qquad \frac{}{((NP\backslash S)\backslash(NP\backslash S))/NP}\ \textsf{about}$$

Using these, prove the following:

(a) *Ivanova flew the Starfury* : S

(b) *Sinclair and Ivanova spoke* : S

(c) *Sinclair spoke about a mission* : S

24. Prove $A\backslash A$ and A/A from no assumptions.

25. $(A \otimes B)\backslash A$ cannot be proven without assumptions. Try to prove it and show where you get stuck.

26. $(A \otimes B)/(B \otimes A)$ cannot be proven without assumptions. Try to prove it and show where you get stuck.

27. Given an assumption of $(A\backslash B)/C$, we can prove $A\backslash(B/C)$, and vice versa. Provide these two proofs.

28. Given an assumption of $A/(B \otimes C)$, we can prove $(A/C)/B$, and vice versa. Provide these two proofs.

3.3 Contexts and Proof Terms

Contexts for the Associative Lambek Calculus are the same as contexts for Intuitionistic Propositional Logic, with the distinction being the types, and the associativity. The types need no explanation — they are just the same sort of types we have been using, with $\backslash$, $/$, and $\otimes$. But the associativity is different. We want to be extremely explicit about it, so we are going to use a new kind of connective, called a **structural connective**. We have already seen one structural connective before: the comma. But it did not do very much for us. Our new connective, which is somewhat like comma, will be $\circ$. It is different from comma precisely in that we lack certain equations of contexts. For comma, we know that $\Gamma, \Delta = \Delta, \Gamma$, which is to say that order does not matter. For $\circ$, order does matter, so we have instead $\Gamma \circ \Delta \neq \Delta \circ \Gamma$. We also lack the ability to "duplicate" contexts, which we will see more clearly in the inference rules. We will also introduce an explicit notation for the empty context — $\mathbf{1}$ — together with the equations $\mathbf{1} \circ \Gamma = \Gamma = \Gamma \circ \mathbf{1}$. Finally, $\circ$ is associative, giving us the equations $(\Gamma \circ \Delta) \circ \Pi = \Gamma \circ (\Delta \circ \Pi)$. We make these equations explicit as (bidirectional) inference rules:

$$\begin{array}{c} \Gamma \vdash \mathcal{X} : X \\ \hline\hline \mathbf{1} \circ \Gamma \vdash \mathcal{X} : X \end{array}\ \textsf{STX-LU}$$

$$\begin{array}{c} \Gamma \vdash \mathcal{X} : X \\ \hline\hline \Gamma \circ \mathbf{1} \vdash \mathcal{X} : X \end{array}\ \textsf{STX-RU}$$

$$\begin{array}{c} (\Gamma \circ \Delta) \circ \Pi \vdash \mathcal{X} : X \\ \hline\hline \Gamma \circ (\Delta \circ \Pi) \vdash \mathcal{X} : X \end{array}\ \textsf{STX-AS}$$

We will also have a new kind of proof term. Rather than simply using an inline notation for rule names, like we did with Intuitionistic Propositional Logic, where, for example, we linearized $\wedge\mathsf{I}$ as a pair $\langle - , - \rangle$, we will instead have proof terms just be words/lexical axioms joined together by $+$. Because the logic is associative, we ought to also expect this string concatenation operator to be associative, so indeed we have equations $(\mathcal{X}+\mathcal{Y})+\mathcal{Z} = \mathcal{X}+(\mathcal{Y}+\mathcal{Z})$. Like contexts, we have an empty proof term (an empty string of words), which satisfies the equations $\epsilon+\mathcal{X} = \mathcal{X} = \mathcal{X}+\epsilon$. These are also expressed via inference rules:

$$\begin{array}{c} \Gamma \vdash \mathcal{X} : X \\ \hline\hline \Gamma \vdash \epsilon+\mathcal{X} : X \end{array}\ \textsf{TM-LU}$$

$$\begin{array}{c} \Gamma \vdash \mathcal{X} : X \\ \hline\hline \Gamma \vdash \mathcal{X}+\epsilon : X \end{array}\ \textsf{TM-RU}$$

$$\begin{array}{c} \Gamma \vdash (\mathcal{X}+\mathcal{Y})+\mathcal{Z} : W \\ \hline\hline \Gamma \vdash \mathcal{X}+(\mathcal{Y}+\mathcal{Z}) : W \end{array}\ \textsf{TM-AS}$$

The introduction and elimination rules for our propositional connectives are now given as

Introduction Rules **Elimination Rules**

$$\dfrac{\Gamma \vdash \mathcal{X} : X \qquad \Delta \vdash \mathcal{Y} : Y}{\Gamma \circ \Delta \vdash \mathcal{X}+\mathcal{Y} : X \otimes Y}\,\otimes\mathsf{I} \qquad \dfrac{\Gamma \vdash \mathcal{P} : X \otimes Y \qquad \Delta[x : X \circ y : Y] \vdash \mathcal{Z}[x+y] : Z}{\Delta[\Gamma] \vdash \mathcal{Z}[\mathcal{P}] : Z}\,\otimes\mathsf{E}$$

$$\dfrac{x : X \circ \Gamma \vdash x+\mathcal{F} : Y}{\Gamma \vdash \mathcal{F} : X\backslash Y}\,\backslash\mathsf{I}_x \qquad \dfrac{\Gamma \vdash \mathcal{X} : X \qquad \Delta \vdash \mathcal{F} : X\backslash Y}{\Gamma \circ \Delta \vdash \mathcal{X}+\mathcal{F} : Y}\,\backslash\mathsf{E}$$

$$\dfrac{\Gamma \circ x : X \vdash \mathcal{F}+x : Y}{\Gamma \vdash \mathcal{F} : Y/X}\,/\mathsf{I}_x \qquad \dfrac{\Gamma \vdash \mathcal{F} : Y/X \qquad \Delta \vdash \mathcal{X} : X}{\Gamma \circ \Delta \vdash \mathcal{F}+\mathcal{X} : Y}\,/\mathsf{E}$$

We also have a hypothesis rule as in Intuitionistic Propositional Logic, but rather than working for arbitrary contexts, it works for the context with only one hypothesis:

$$\dfrac{}{x : X \vdash x : X}\,\mathsf{hyp}$$

3.3.1 Examples

We can now give a proof that *Sinclair saw Ivanova* : S. However, since we have proof terms, we can be precise, and say instead that it is a proof of $\mathbf{1} \vdash \mathsf{Sinclair}+(\mathsf{saw}+\mathsf{Ivanova}) : S$. The lexical axioms that we need to prove this are

$$\dfrac{}{\mathbf{1} \vdash \mathsf{Sinclair} : NP}\,\mathsf{ax} \qquad \dfrac{}{\mathbf{1} \vdash \mathsf{Ivanova} : NP}\,\mathsf{ax} \qquad \dfrac{}{\mathbf{1} \vdash \mathsf{saw} : (NP\backslash S)/NP}\,\mathsf{ax}$$

and the proof is

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathsf{Sinclair} : NP}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{\mathbf{1} \vdash \mathsf{saw} : (NP\backslash S)/NP}\,\mathsf{ax} \qquad \dfrac{}{\mathbf{1} \vdash \mathsf{Ivanova} : NP}\,\mathsf{ax}}{\mathbf{1} \circ \mathbf{1} \vdash \mathsf{saw}+\mathsf{Ivanova} : NP\backslash S}\,/\mathsf{E}}{\mathbf{1} \circ (\mathbf{1} \circ \mathbf{1}) \vdash \mathsf{Sinclair}+(\mathsf{saw}+\mathsf{Ivanova}) : S}\,\backslash\mathsf{E}}{\mathbf{1} \circ \mathbf{1} \vdash \mathsf{Sinclair}+(\mathsf{saw}+\mathsf{Ivanova}) : S}\,\mathsf{STX\text{-}LU}}{\mathbf{1} \vdash \mathsf{Sinclair}+(\mathsf{saw}+\mathsf{Ivanova}) : S}\,\mathsf{STX\text{-}LU}$$

Instead of using STX-LU twice, we could have used STX-RU the second time, or we could have used STX-AS first, and so forth. Because it is often obvious what structural rules are necessary, we can abbreviate this proof like so:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\mathbf{1} \vdash \mathsf{Sinclair} : NP$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\mathbf{1} \vdash \mathsf{saw} : (NP\backslash S)/NP$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\mathbf{1} \vdash \mathsf{Ivanova} : NP$}
\RightLabel{$/\mathsf{E}$}
\BinaryInfC{$\mathbf{1}\circ\mathbf{1} \vdash \mathsf{saw}+\mathsf{Ivanova} : NP\backslash S$}
\RightLabel{$\backslash\mathsf{E}$}
\BinaryInfC{$\mathbf{1}\circ(\mathbf{1}\circ\mathbf{1}) \vdash \mathsf{Sinclair}+(\mathsf{saw}+\mathsf{Ivanova}) : S$}
\dashedLine
\RightLabel{STX}
\UnaryInfC{$\mathbf{1} \vdash \mathsf{Sinclair}+(\mathsf{saw}+\mathsf{Ivanova}) : S$}
\end{prooftree}
$$

An analogous abbreviation can be done for TM-LU, TM-RU, and TM-AS, abbreviating to TM with a dashed line. We can see these term rules in action, by doing the version of this proof with a detour:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\mathbf{1} \vdash \mathsf{Sinclair} : NP$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\mathbf{1} \vdash \mathsf{saw} : (NP\backslash S)/NP$}
\AxiomC{}
\RightLabel{hyp}
\UnaryInfC{$x : NP \vdash x : NP$}
\RightLabel{$/\mathsf{E}$}
\BinaryInfC{$\mathbf{1}\circ x : NP \vdash \mathsf{saw}+x : NP\backslash S$}
\RightLabel{$\backslash\mathsf{E}$}
\BinaryInfC{$\mathbf{1}\circ(\mathbf{1}\circ x : NP) \vdash \mathsf{Sinclair}+(\mathsf{saw}+x) : S$}
\RightLabel{STX-LU}
\UnaryInfC{$\mathbf{1}\circ x : NP \vdash \mathsf{Sinclair}+(\mathsf{saw}+x) : S$}
\RightLabel{TM-AS}
\UnaryInfC{$\mathbf{1}\circ x : NP \vdash (\mathsf{Sinclair}+\mathsf{saw})+x : S$}
\RightLabel{$/\mathsf{I}_x$}
\UnaryInfC{$\mathbf{1} \vdash \mathsf{Sinclair}+\mathsf{saw} : S/NP$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\mathbf{1} \vdash \mathsf{Ivanova} : NP$}
\RightLabel{$/\mathsf{E}$}
\BinaryInfC{$\mathbf{1}\circ\mathbf{1} \vdash (\mathsf{Sinclair}+\mathsf{saw})+\mathsf{Ivanova} : S$}
\RightLabel{STX-RU}
\UnaryInfC{$\mathbf{1} \vdash (\mathsf{Sinclair}+\mathsf{saw})+\mathsf{Ivanova} : S$}
\RightLabel{TM-AS}
\UnaryInfC{$\mathbf{1} \vdash \mathsf{Sinclair}+(\mathsf{saw}+\mathsf{Ivanova}) : S$}
\end{prooftree}
$$

Again, the presence of all these structural and term rules warrants a combined STX and TM notation, which we just indicate with a comma, like so:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathsf{Sinclair} : NP}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{\mathbf{1} \vdash \mathsf{saw} : (NP\backslash S)/NP}\,\mathsf{ax} \quad \dfrac{}{x : NP \vdash x : NP}\,\mathsf{hyp}}{\mathbf{1} \circ x : NP \vdash \mathsf{saw}+x : NP\backslash S}\,/\mathsf{E}}{\mathbf{1} \circ (\mathbf{1} \circ x : NP) \vdash \mathsf{Sinclair}+(\mathsf{saw}+x) : S}\,\backslash\mathsf{E}}{\mathbf{1} \circ x : NP \vdash (\mathsf{Sinclair}+\mathsf{saw})+x : S}\,\mathsf{STX,TM}}{\mathbf{1} \vdash \mathsf{Sinclair}+\mathsf{saw} : S/NP}\,/\mathsf{I}_x \qquad \dfrac{}{\mathbf{1} \vdash \mathsf{Ivanova} : NP}\,\mathsf{ax}}{\dfrac{\mathbf{1} \circ \mathbf{1} \vdash (\mathsf{Sinclair}+\mathsf{saw})+\mathsf{Ivanova} : S}{\mathbf{1} \vdash \mathsf{Sinclair}+(\mathsf{saw}+\mathsf{Ivanova}) : S}\,\mathsf{STX,TM}}\,/\mathsf{E}
$$

This notation, of listing rules together on a single inference rather than showing each inference, is quite common for "obvious" proofs.

3.3.2 Exercises

29. Re-do the proofs from 3.2.2 using contexts and proof terms.

3.4 Sequent Calculus

The Sequent Calculus version of the Associative Lambek Calculus can now be fairly straightforwardly arrived at by analogy to the Sequent Calculus for Intuitionistic Propositional Logic, but with the relevant to the structure of the context. The right rules are, as before, the same as introductions.

Right Rules

$$
\dfrac{\Gamma \Rightarrow \mathcal{M} : X \qquad \Delta \Rightarrow \mathcal{N} : Y}{\Gamma \circ \Delta \Rightarrow \mathcal{M}+\mathcal{N} : X \otimes Y}\,\otimes\mathsf{R}
$$

$$
\dfrac{x : X \circ \Gamma \Rightarrow x+\mathcal{M} : Y}{\Gamma \Rightarrow \mathcal{M} : X\backslash Y}\,\backslash\mathsf{R}
$$

$$
\dfrac{\Gamma \circ x : X \Rightarrow \mathcal{M}+x : Y}{\Gamma \Rightarrow \mathcal{M} : Y/X}\,/\mathsf{R}
$$

Left Rules

$$
\dfrac{\Gamma[x : X \circ y : Y] \Rightarrow \mathcal{M}[x+y] : Z}{\Gamma[p : X \otimes Y] \Rightarrow \mathcal{M}[p]}\,\otimes\mathsf{L}
$$

$$
\dfrac{\Gamma \Rightarrow \mathcal{M} : X \qquad \Delta[y : Y] \Rightarrow \mathcal{N}[y] : Z}{\Delta[\Gamma \circ f : X\backslash Y] \Rightarrow \mathcal{N}[\mathcal{M}+f] : Z}\,\backslash\mathsf{L}
$$

$$
\dfrac{\Gamma \Rightarrow \mathcal{M} : X \qquad \Delta[y : Y] \Rightarrow \mathcal{N}[y] : Z}{\Delta[f : Y/X \circ \Gamma] \Rightarrow \mathcal{N}[f+\mathcal{M}] : Z}\,/\mathsf{L}
$$

As with the previous Sequent Calculus system, we will also have an ID rule and a CUT rule:

$$\begin{array}{c} \\ \hline x : X \ \Rightarrow\ x : X \end{array}\ \textsf{ID}$$

$$\begin{array}{c} \Gamma \ \Rightarrow\ \mathcal{M} : X \qquad \Delta[x : X] \ \Rightarrow\ \mathcal{N}[x] : Y \\ \hline \Delta[\Gamma] \ \Rightarrow\ \mathcal{N}[\mathcal{M}] : Y \end{array}\ \textsf{CUT}$$

We also carry over the structural and term rules unchanged, except for notation:

$$\begin{array}{c} \Gamma \ \Rightarrow\ \mathcal{X} : X \\ \hline\hline \mathbf{1} \circ \Gamma \ \Rightarrow\ \mathcal{X} : X \end{array}\ \textsf{STX-LU}$$

$$\begin{array}{c} \Gamma \ \Rightarrow\ \mathcal{X} : X \\ \hline\hline \Gamma \circ \mathbf{1} \ \Rightarrow\ \mathcal{X} : X \end{array}\ \textsf{STX-RU}$$

$$\begin{array}{c} (\Gamma \circ \Delta) \circ \Pi \ \Rightarrow\ \mathcal{X} : X \\ \hline\hline \Gamma \circ (\Delta \circ \Pi) \ \Rightarrow\ \mathcal{X} : X \end{array}\ \textsf{STX-AS}$$

$$\begin{array}{c} \Gamma \ \Rightarrow\ \mathcal{X} : X \\ \hline\hline \Gamma \ \Rightarrow\ \epsilon + \mathcal{X} : X \end{array}\ \textsf{TM-LU}$$

$$\begin{array}{c} \Gamma \ \Rightarrow\ \mathcal{X} : X \\ \hline\hline \Gamma \ \Rightarrow\ \mathcal{X} + \epsilon : X \end{array}\ \textsf{TM-RU}$$

$$\begin{array}{c} \Gamma \ \Rightarrow\ (\mathcal{X} + \mathcal{Y}) + \mathcal{Z} : W \\ \hline\hline \Gamma \ \Rightarrow\ \mathcal{X} + (\mathcal{Y} + \mathcal{Z}) : W \end{array}\ \textsf{TM-AS}$$

3.4.1 Examples

Lexical axioms in this system are essentially the same as before, again except for notation:

$$\begin{array}{c} \\ \hline \mathbf{1} \ \Rightarrow\ \textsf{Sinclair} : NP \end{array}\ \textsf{ax} \qquad \begin{array}{c} \\ \hline \mathbf{1} \ \Rightarrow\ \textsf{Ivanova} : NP \end{array}\ \textsf{ax} \qquad \begin{array}{c} \\ \hline \mathbf{1} \ \Rightarrow\ \textsf{saw} : (NP \backslash S)/NP \end{array}\ \textsf{ax}$$

The proof of that *Sinclair saw Susan* is a sentence now becomes

$$\text{lemma}_1 = \dfrac{\dfrac{}{y : NP \;\Rightarrow\; y : NP}\,\mathsf{ID} \qquad \dfrac{\dfrac{}{x : NP \;\Rightarrow\; x : NP}\,\mathsf{ID} \qquad \dfrac{}{s : S \;\Rightarrow\; s : S}\,\mathsf{ID}}{x : NP \circ g : NP\backslash S \;\Rightarrow\; x{+}g : S}\,\backslash\mathsf{L}}{x : NP \circ (f : (NP\backslash S)/NP \circ y : NP) \;\Rightarrow\; x{+}(f{+}y) : S}\,/\mathsf{L}$$

$$\dfrac{\dfrac{\dfrac{}{\mathbf{1} \;\Rightarrow\; \mathsf{Ivanova} : NP}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{\mathbf{1} \;\Rightarrow\; \mathsf{saw} : (NP\backslash S)/NP}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{\mathbf{1} \;\Rightarrow\; \mathsf{Sinclair} : NP}\,\mathsf{ax} \qquad \text{lemma}_1}{\mathbf{1} \circ (f : (NP\backslash S)/NP \circ y : NP) \;\Rightarrow\; \mathsf{Sinclair}{+}(f{+}y) : S}\,\mathsf{CUT}}{\mathbf{1} \circ (\mathbf{1} \circ y : NP) \;\Rightarrow\; \mathsf{Sinclair}{+}(\mathsf{saw}{+}y) : S}\,\mathsf{CUT}}{\mathbf{1} \circ (\mathbf{1} \circ \mathbf{1}) \;\Rightarrow\; \mathsf{Sinclair}{+}(\mathsf{saw}{+}\mathsf{Ivanova}) : S}\,\mathsf{CUT}}{\mathbf{1} \;\Rightarrow\; \mathsf{Sinclair}{+}(\mathsf{saw}{+}\mathsf{Ivanova}) : S}\,\mathsf{STX}$$

A typical proof begins by cutting out all of the lexical items in this fashion, so we can abbreviate these cuts against lexical axioms by LEX like so:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y : NP \;\Rightarrow\; y : NP}\,\mathsf{ID} \qquad \dfrac{\dfrac{}{x : NP \;\Rightarrow\; x : NP}\,\mathsf{ID} \qquad \dfrac{}{s : S \;\Rightarrow\; s : S}\,\mathsf{ID}}{x : NP \circ g : NP\backslash S \;\Rightarrow\; x{+}g : S}\,\backslash\mathsf{L}}{x : NP \circ (f : (NP\backslash S)/NP \circ y : NP) \;\Rightarrow\; x{+}(f{+}y) : S}\,/\mathsf{L}}{\mathbf{1} \circ (\mathbf{1} \circ \mathbf{1}) \;\Rightarrow\; \mathsf{Sinclair}{+}(\mathsf{saw}{+}\mathsf{Ivanova}) : S}\,\mathsf{LEX}}{\mathbf{1} \;\Rightarrow\; \mathsf{Sinclair}{+}(\mathsf{saw}{+}\mathsf{Ivanova}) : S}\,\mathsf{STX}$$

When working bottom up, as one does with Sequent Calculus, it is important to cut against the right components of the context, and to use the right structural rules. Because we start out with just the unit context $\mathbf{1}$, we can expand it to whatever we want using the structural rules, and then cut along those in whatever way we choose, but the result will not always be useful. For instance, we could have expanded and cut like so:

$$\dfrac{\dfrac{\dfrac{\vdots}{(f : (NP\backslash S)/NP \circ y : NP) \circ x : NP \;\Rightarrow\; x{+}(f{+}y) : S}}{(\mathbf{1} \circ \mathbf{1}) \circ \mathbf{1} \;\Rightarrow\; \mathsf{Sinclair}{+}(\mathsf{saw}{+}\mathsf{Ivanova}) : S}\,\mathsf{LEX}}{\mathbf{1} \;\Rightarrow\; \mathsf{Sinclair}{+}(\mathsf{saw}{+}\mathsf{Ivanova}) : S}\,\mathsf{STX}$$

But this would get stuck fairly quickly and we would have to start over. In general, we only have to expand the context to contain as many $\mathbf{1}$ occurrences as there are words in the sentence, so that at least limits the choices we can make. Typically, the order of the hypotheses corresponds to their order in the proof term. but these are both heuristics, not guarantees.

3.4.2 Exercises

30. Re-do the proofs from 3.2.2 using the Sequent Calculus

31. Each of the following sequents can be derived from each of the others. Provide the relevant proofs. You should provide six, one for each pair of sequents and each direction of proof.

    (a) $A \otimes B \;\Rightarrow\; C$

    (b) $B \;\Rightarrow\; A\backslash C$

    (c) $A \;\Rightarrow\; C/B$

3.5 Semantics

Assigning a semantics to the expressions of the Associative Lambek Calculus can be done relatively easily by way of parallel Intuitionistic Propositional Logic proofs. We will need some extra types to do this, however. The first extra type is relatively simple: the type of boolean values B together with its two proofs/inhabitants yes and no. We also need a second extra type, E, for entities like Sinclair and Ivanova. The proofs/inhabitants of E will be given as needed, usually in axioms. Axioms and inference rules get expanded to run both proofs simultaneously. Before giving the new inference rules, which are relatively obvious, consider the following example proof for the sentence *Sinclair saw Ivanova*:

$$
\begin{array}{c}
\dfrac{
\dfrac{}{\begin{array}{c} \mathbf{1} \vdash \mathsf{Sinclair} : NP \\ \varnothing \vdash \mathsf{s} : E \end{array}}\,\mathsf{ax}
\qquad
\dfrac{
\dfrac{}{\begin{array}{c} \mathbf{1} \vdash \mathsf{saw} : (NP\backslash S)/NP \\ \varnothing \vdash \mathsf{see} : E \to E \to B \end{array}}\,\mathsf{ax}
\qquad
\dfrac{}{\begin{array}{c} \mathbf{1} \vdash \mathsf{Ivanova} : NP \\ \varnothing \vdash \mathsf{i} : E \end{array}}\,\mathsf{ax}
}{\begin{array}{c} \mathbf{1} \circ \mathbf{1} \vdash \mathsf{saw}+\mathsf{Ivanova} : NP\backslash S \\ \varnothing, \varnothing \vdash \mathsf{see\ i} : E \to B \end{array}}\,\begin{array}{l} /\mathsf{E} \\ \to\mathsf{E} \end{array}
}{\begin{array}{c} \mathbf{1} \circ (\mathbf{1} \circ \mathbf{1}) \vdash \mathsf{Sinclair}+(\mathsf{saw}+\mathsf{Ivanova}) : S \\ \varnothing, (\varnothing, \varnothing) \vdash \mathsf{see\ i\ s} : B \end{array}}\,\begin{array}{l} \backslash\mathsf{E} \\ \to\mathsf{E} \end{array} \\
\hdashline
\begin{array}{c} \mathbf{1} \vdash \mathsf{Sinclair}+(\mathsf{saw}+\mathsf{Ivanova}) : S \\ \varnothing \vdash \mathsf{see\ i\ s} : B \end{array}
\end{array}
\;\mathsf{STX}
$$

Notice that the contexts for the intuitionistic component are now explicitly manipulated, whereas previously they were only implicitly manipulated before. Also notice that there is twice as much to prove. We will abbreviate these double proofs to make this less of a hassle. Rather than writing simultaneous proofs like this:

$$\begin{array}{r c l} x : X \circ y : Y & \vdash & \mathscr{Z} : Z \\ x' : X', y' : Y' & \vdash & \mathscr{Z}' : Z' \end{array}$$

We will omit the intuitionistic portions, except for the proof terms/semantics, which will be written next to the Lambek proof terms separated by a line, like so:

$$x - x' : X \circ y - y' : Y \vdash \mathscr{Z} - \mathscr{Z}' : Z$$

Using this new notation, the proof for *Sinclair saw Ivanova* would be

$$\dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathsf{Sinclair} - \mathsf{s} : NP}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{\mathbf{1} \vdash \mathsf{saw} - \mathsf{see} : (NP\backslash S)/NP}\,\mathsf{ax} \qquad \dfrac{}{\mathbf{1} \vdash \mathsf{Ivanova} - \mathsf{i} : NP}\,\mathsf{ax}}{\mathbf{1} \circ \mathbf{1} \vdash \mathsf{saw}{+}\mathsf{Ivanova} - \mathsf{see\ i} : NP\backslash S}\,/\mathsf{E}}{\mathbf{1} \circ (\mathbf{1} \circ \mathbf{1}) \vdash \mathsf{Sinclair}{+}(\mathsf{saw}{+}\mathsf{Ivanova}) - \mathsf{see\ i\ s} : S}\,\backslash\mathsf{E}}{\mathbf{1} \vdash \mathsf{Sinclair}{+}(\mathsf{saw}{+}\mathsf{Ivanova}) - \mathsf{see\ i\ s} : S}\,\mathsf{STX}$$

To do this, however, we have to be careful that the semantic terms have the appropriate types — it is no good writing abbreviated proofs like this if the semantic portion is not actually a valid proof. For types like NP and S, this is not much of a problem because they correspond directly to atomic types in the semantic component, but for types like N, it is usually taken to correspond to a type such as $E \to B$, because the meaning of a noun like *Starfury* is a predicate that maps entities to yes if the entity is a Starfury, and to no if it is not a Starfury.

The inference rules for the Associative Lambek Calculus with semantics are given in abbreviated form as

Introduction Rules

$$\frac{\Gamma \;\vdash\; \mathcal{X} - \mathcal{X}' : X \qquad \Delta \;\vdash\; \mathcal{Y} - \mathcal{Y}' : Y}{\Gamma \circ \Delta \;\vdash\; \mathcal{X} + \mathcal{Y} - \langle \mathcal{X}' , \mathcal{Y}' \rangle : X \otimes Y}\ \otimes\mathsf{I}$$

$$\frac{x - x' : X \circ \Gamma \;\vdash\; x + \mathcal{F} - \mathcal{F}' : Y}{\Gamma \;\vdash\; \mathcal{F} - \lambda x'.\mathcal{F}' : X \backslash Y}\ \backslash\mathsf{I}_x$$

$$\frac{\Gamma \circ x - x' : X \;\vdash\; \mathcal{F} + x - \mathcal{F}' : Y}{\Gamma \;\vdash\; \mathcal{F} - \lambda x'.\mathcal{F}' : Y / X}\ /\mathsf{I}_x$$

Elimination Rules

$$\frac{\Gamma \;\vdash\; \mathcal{P} - \mathcal{P}' : X \otimes Y \qquad \Delta[x - x' : X \circ y - y' : Y] \;\vdash\; \mathcal{Z}[x + y] - \mathcal{Z}' : Z}{\Delta[\Gamma] \;\vdash\; \mathcal{Z}[\mathcal{P}] - \mathcal{Z}'[\mathsf{fst}\ \mathcal{P}'/x, \mathsf{snd}\ \mathcal{P}'/y] : Z}\ \otimes\mathsf{E}$$

$$\frac{\Gamma \;\vdash\; \mathcal{X} - \mathcal{X}' : X \qquad \Delta \;\vdash\; \mathcal{F} - \mathcal{F}' : X \backslash Y}{\Gamma \circ \Delta \;\vdash\; \mathcal{X} + \mathcal{F} - \mathcal{F}'\,\mathcal{X}' : Y}\ \backslash\mathsf{E}$$

$$\frac{\Gamma \;\vdash\; \mathcal{F} - \mathcal{F}' : Y / X \qquad \Delta \;\vdash\; \mathcal{X} - \mathcal{X}' : X}{\Gamma \circ \Delta \;\vdash\; \mathcal{F} + \mathcal{X} - \mathcal{F}'\,\mathcal{X}' : Y}\ /\mathsf{E}$$

The hypothesis rule is as expected:

$$\frac{}{x - x' : X \;\vdash\; x - x' : X}\ \mathsf{hyp}$$

The structural rules are nearly identical identical, but with extra proof terms for the semantics:

$$\begin{array}{c} \Gamma \;\vdash\; \mathcal{X} - \mathcal{X}' : X \\ \hline\hline \mathbf{1} \circ \Gamma \;\vdash\; \mathcal{X} - \mathcal{X}' : X \end{array}\ \mathsf{STX\text{-}LU}$$

$$\begin{array}{c} \Gamma \;\vdash\; \mathcal{X} - \mathcal{X}' : X \\ \hline\hline \Gamma \circ \mathbf{1} \;\vdash\; \mathcal{X} - \mathcal{X}' : X \end{array}\ \mathsf{STX\text{-}RU}$$

$$\begin{array}{c} (\Gamma \circ \Delta) \circ \Pi \;\vdash\; \mathcal{X} - \mathcal{X}' : X \\ \hline\hline \Gamma \circ (\Delta \circ \Pi) \;\vdash\; \mathcal{X} - \mathcal{X}' : X \end{array}\ \mathsf{STX\text{-}AS}$$

While the term rules have the same effect on the syntax but no effect on the semantics:

$$\begin{array}{c} \Gamma \;\vdash\; \mathcal{X} - \mathcal{X}' : X \\ \hline\hline \Gamma \;\vdash\; \epsilon + \mathcal{X} - \mathcal{X}' : X \end{array}\ \mathsf{TM\text{-}LU}$$

$$\begin{array}{c} \Gamma \;\vdash\; \mathcal{X} - \mathcal{X}' : X \\ \hline\hline \Gamma \;\vdash\; \mathcal{X} + \epsilon - \mathcal{X}' : X \end{array}\ \mathsf{TM\text{-}RU}$$

$$\begin{array}{c} \Gamma \;\vdash\; (\mathcal{X} + \mathcal{Y}) + \mathcal{Z} - \mathcal{X}' : W \\ \hline\hline \Gamma \;\vdash\; \mathcal{X} + (\mathcal{Y} + \mathcal{Z}) - \mathcal{X}' : W \end{array}\ \mathsf{TM\text{-}AS}$$