Midterm

Due 21 October

During proof search, we usually have lots of options. Therefore any way to guide the proof search and make some options “as good” as any other is going to be of some benefit. To show that two options — two applicable inference rules — are as good as one another, we have to show that it doesn’t matter which order they are applied in. If two rules, R_1 and R_2, can be applied in either order without changing provability, then we say these rules commute. For example, $\wedge\mathsf{L}^{\mathsf{L}}$ and $\wedge\mathsf{L}^{\mathsf{R}}$. We can show this relatively easily. Given any portion of a proof that looks like this:

$$\cfrac{\cfrac{\Gamma,\ p : A \wedge B,\ x : A, y : B \ \Rightarrow\ \square : C}{\Gamma,\ p : A \wedge B,\ x : A \ \Rightarrow\ \square : C}\ \wedge\mathsf{L}^{\mathsf{R}}}{\Gamma,\ p : A \wedge B \ \Rightarrow\ \square : C}\ \wedge\mathsf{L}^{\mathsf{L}}$$

we can convert it to a related proof, applying the inference rules in the other order, but still starting and ending up in at the same sequents:

$$\cfrac{\cfrac{\Gamma,\ p : A \wedge B,\ x : A, y : B \ \Rightarrow\ \square : C}{\Gamma,\ p : A \wedge B,\ y : B \ \Rightarrow\ \square : C}\ \wedge\mathsf{L}^{\mathsf{L}}}{\Gamma,\ p : A \wedge B \ \Rightarrow\ \square : C}\ \wedge\mathsf{L}^{\mathsf{R}}$$

The conversion in the opposite direction works as well, thereby showing that they do in fact permute.

We can also show that $\wedge\mathsf{L}^{\mathsf{L}}$ and $\wedge\mathsf{R}$ permute, because the following two proofs have identical final sequent at the bottom, and identical goal sequents at the tops (albeit repeated):

$$\dfrac{\dfrac{\Gamma,\ p : A \wedge B,\ x : A \ \Rightarrow\ \square : C \qquad \Gamma,\ p : A \wedge B,\ x : A \ \Rightarrow\ \square : D}{\Gamma,\ p : A \wedge B,\ x : A \ \Rightarrow\ \square : C \wedge D}\ \wedge\mathsf{R}}{\Gamma,\ p : A \wedge B \ \Rightarrow\ \square : C \wedge D}\ \wedge\mathsf{L}^{\mathsf{L}}$$

$$\dfrac{\dfrac{\Gamma,\ p : A \wedge B,\ x : A \ \Rightarrow\ \square : C}{\Gamma,\ p : A \wedge B \ \Rightarrow\ \square : C}\ \wedge\mathsf{L}^{\mathsf{L}} \qquad \dfrac{\Gamma,\ p : A \wedge B,\ x : A \ \Rightarrow\ \square : D}{\Gamma,\ p : A \wedge B \ \Rightarrow\ \square : D}\ \wedge\mathsf{L}^{\mathsf{L}}}{\Gamma,\ p : A \wedge B \ \Rightarrow\ \square : C \wedge D}\ \wedge\mathsf{R}$$

A nearly identical proof can be constructed for $\wedge\mathsf{L}^{\mathsf{R}}$. What this shows, then, is that the inference rules for $\wedge$ can be applied in any order, relative to one another, and the choice doesn't matter, because you can always end up with the same proof obligations as any other choice.

Similarly, we can freely permute $\wedge\mathsf{L}^{\mathsf{L}}$ (resp. $\wedge\mathsf{L}^{\mathsf{R}}$) and $\to\mathsf{L}$:

$$\dfrac{\dfrac{\Gamma,\ f : A \to B,\ p : C \wedge D,\ z : C \ \Rightarrow\ \square : A \qquad \Gamma,\ f : A \to B,\ y : B,\ p : C \wedge D,\ z : C \ \Rightarrow\ \square : E}{\Gamma,\ f : A \to B,\ p : C \wedge D,\ z : C \ \Rightarrow\ \square : E}\ \to\mathsf{L}_{f,y}}{\Gamma,\ f : A \to B,\ p : C \wedge D \ \Rightarrow\ \square : E}\ \wedge\mathsf{L}^{\mathsf{L}}{}_{p,z}$$

$$\dfrac{\dfrac{\Gamma,\ f : A \to B,\ p : C \wedge D,\ z : C \ \Rightarrow\ \square : A}{\Gamma,\ f : A \to B,\ p : C \wedge D \ \Rightarrow\ \square : A}\ \wedge\mathsf{L}^{\mathsf{L}}{}_{p,z} \qquad \dfrac{\Gamma,\ f : A \to B,\ y : B,\ p : C \wedge D,\ z : C \ \Rightarrow\ \square : E}{\Gamma,\ f : A \to B,\ y : B,\ p : C \wedge D \ \Rightarrow\ \square : E}\ \wedge\mathsf{L}^{\mathsf{L}}{}_{p,z}}{\Gamma,\ f : A \to B,\ p : C \wedge D \ \Rightarrow\ \square : E}\ \to\mathsf{L}_{f,y}$$

Problems on next page.

1 Problem 1

Show that $\wedge\mathsf{R}$ and $\to\mathsf{L}$ permute.

2 Problem 2

Show that $\wedge\mathsf{L}^{\mathsf{L}}$ and $\to\mathsf{R}$ permute.

3 Problem 3

Show that $\to\mathsf{R}$ and $\to\mathsf{L}$ permute.

4 Problem 4

Permutation doesn't always work: the following proof cannot have it's $\to\mathsf{L}$ and $\to\mathsf{R}$ permuted. Explain why not, and give a general statement about when the order of rules matters.

$$\dfrac{\dfrac{\Gamma,\ f : A \to B \ \Rightarrow\ \square : A \qquad \Gamma,\ f : A \to B,\ y : B \ \Rightarrow\ \square : C}{\Gamma,\ f : A \to B \ \Rightarrow\ \square : C}\ {\to}\mathsf{L}_{f,y}}{\Gamma \ \Rightarrow\ \square : (A \to B) \to C}\ {\to}\mathsf{R}_{f}$$